Synthesizing Computable Functions from Synchronous Specifications

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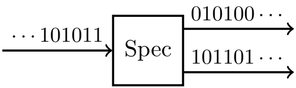
YR-OWLS, online

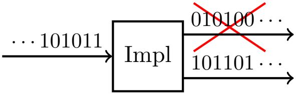
Reactive Synthesis of Non-terminating Systems

$$\textbf{Specification} \overset{\text{synthesize}}{\dashrightarrow} \textbf{Implementation}$$

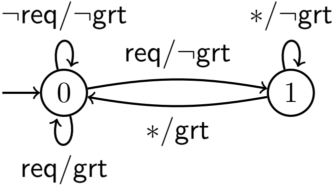
one input is in relation
with several outputs

algorithm that selects
a unique output for each input





Church Synthesis



Synchronous specifications
(**synchronous relations**)
e.g, given by
synchronous transducers with
parity acceptance

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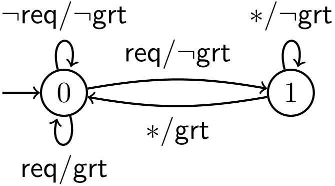

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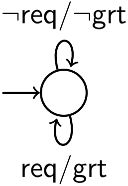

Synchronous implementations
given by
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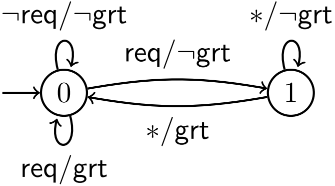

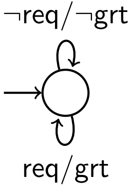

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Synchronous implementations
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Theorem (Büchi/Landweber'69). It is decidable whether a synchronous specification is implementable by a Mealy machine.

More Relaxed Implementations

Goal Decide whether a synchronous specification is implementable (by an algorithm/a program/a deterministic Turing machine).

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- can be implemented, every deterministic machine has to wait until it sees the third input letter

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$$(uA\alpha, A^{|u|}\beta) \quad (uB\alpha, B^{|u|}\beta),$$

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- e.g., implemented by a deterministic machine that computes the function

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A function $f \colon \Sigma^\omega \rightharpoonup \Gamma^\omega$ is **computable** if there exists a deterministic Turing machine that

- outputs longer and longer prefixes of an acceptable output
- while it reads longer and longer prefixes of the input.

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M **computes** f if for all $\alpha \in \mathrm{dom}(f)$:

- $\forall k$: $M(\alpha, k)$ is a prefix of $f(\alpha)$, and
- $\forall i \; \exists j$: $|M(\alpha, j)| \geq i$

Computability and Continuity

Computability and Continuity

A function $f\colon \Sigma^\omega \rightharpoonup \Gamma^\omega$ is **continuous** at $\alpha \in \mathrm{dom}(f)$ if

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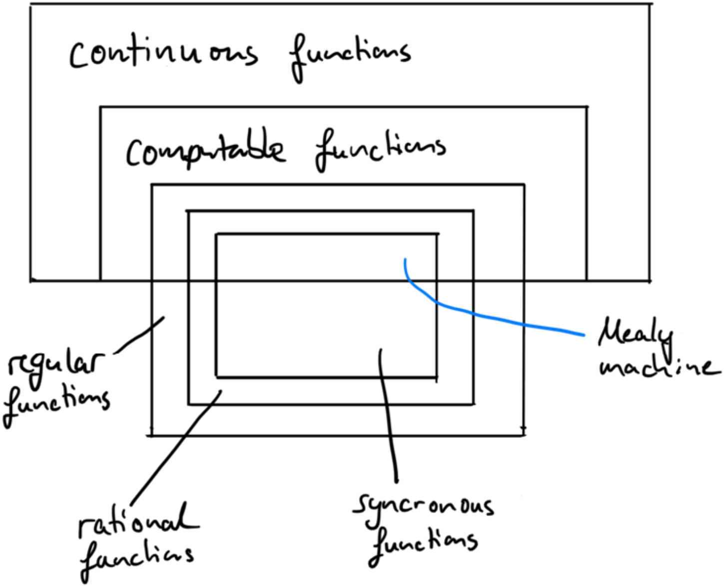
- If $f\colon \Sigma^\omega \rightharpoonup \Gamma^\omega$ is computable, then it is continuous,
- the converse does not hold.

Computability and Continuity



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- There is no way to complete the domain and remain implementable!

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Is the function computable?

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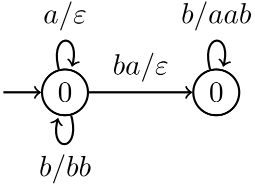

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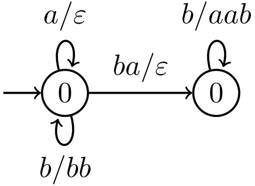

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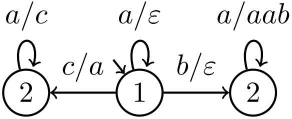

sequential transducer

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Theorem (Filiot/W.). It is EXPTIME-complete to decide whether a continuous function can be synthesized from a given synchronous relation with **partial domain**. Such a synthesized function is computable.

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Game view

- Adam plays input letters

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Solution

- Instead of an explicit lookahead, store a finite abstraction

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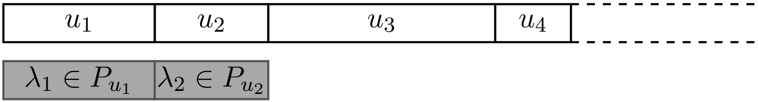

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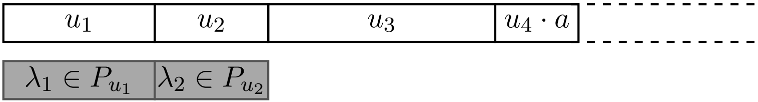

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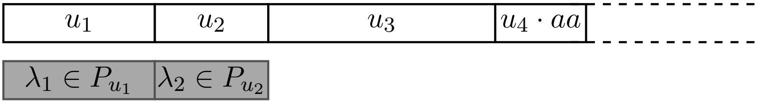

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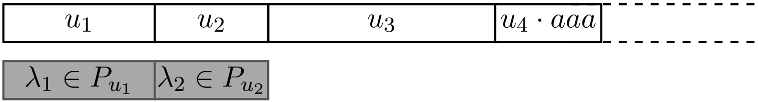

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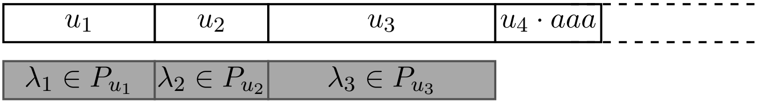

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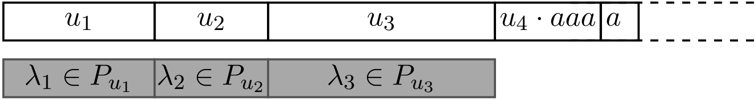

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- Eves makes a move infinitely often,
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  - goes right and copies the input

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  - If an arbitrary long lookahead is needed to determine the next output,

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- Finite word setting: Undecidable whether a sequential function can be synthesized. (Carayol/Löding'14)

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PCP instance has a solution

- no implementation exists
- never known whether the input sequence has ∞ many a

Work in Progress: Deterministic Rational Relations

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Class between synchronous and rational relations.

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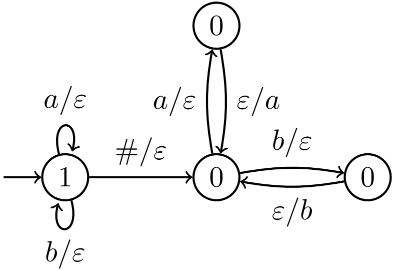

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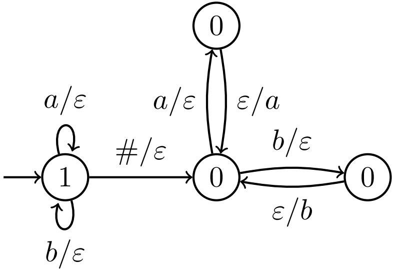

- recognizes $f\colon u\#\alpha \mapsto \alpha, \quad u \in \{a,b\}^*, \alpha \in \{a,b\}^\omega$
- f is not synchronous

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Almost Sure Theorem. Such a synthesized function is computable by a deterministic two-way transducer.

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Example.

- Specification: $(a^*b\cdots, b\cdots)$ $(a^*c\cdots, c\cdots)$
- Specification is implementable, e.g., by a finite-memory machine (sequential transducer) that computes the function

$$a^*b\cdots \mapsto b^\omega \quad a^*c\cdots \mapsto c^\omega$$

Summary

Spec \ Impl	Mealy machine	sequential transducer	computable
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synchronous w/ partial domain	EXPTIME-c[1]	open	EXPTIME-c[2]
det. rational	open	open	EXPTIME-c
rational	undecidable	undecidable	undecidable

[1] non-deterministic specification [2] deterministic specification