# Relation algebra
# Decidability & Axiomatizability

YR-OWLS

July $1^{st}$ 2020

Amina Doumane
CNRS- ENS Lyon

# Binary relations are everywhere

- **Graph theory**
- **Semantics of imperative programs**
- **Foundations of mathematics**

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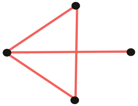

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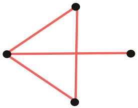

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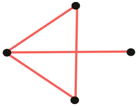

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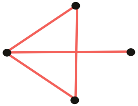

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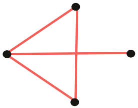

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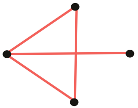

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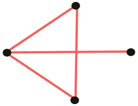

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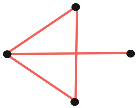

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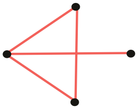

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- **Foundations of mathematics**

| | |
|---|---|
| Two binary relations | $\epsilon$ (membership), $1$ (identity) |
| Operations | $\cup$ (union), $\cdot$ (composition), $\smile$ (converse), $^c$ (complement) |
| Sentences | $e = f$ |

# Relation algebra

## Relational Operators

| | | |
|---|---|---|
| identity relation | : | $1$ |
| empty relation | : | $0$ |
| composition | : | $R \cdot S$ |
| union | : | $R \cup S$ |
| intersection | : | $R \cap S$ |
| trans. closure | : | $R^+$ |
| converse | : | $R^\smile$ |
| complement | : | $R^c$ |

# Relation algebra and their universal laws

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$$\mathrm{Rel} \models R \cdot (S \cdot R)^+ = (R \cdot S)^+ \cdot R$$

$$\mathrm{Rel} \models 1 \cup R^\star \cdot S \subseteq (R \cup S)^\star$$

$$\mathrm{Rel} \models (R \cap S) \cdot T \subseteq (R \cdot T) \cap (S \cdot T)$$

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**Decidability** and **Axiomatizability**

# Deciding the equational theory of Relation Algebra

## Decidability problem

**Input:** Expressions $e$ and $f$.

**Output:** Is $\mathrm{Rel} \models e = f$ a universal law?

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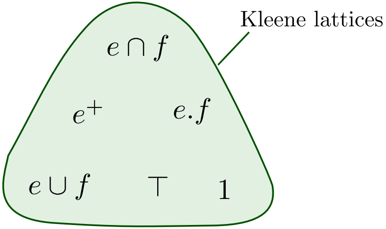

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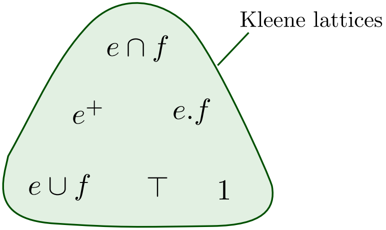


EXPSPACE-complete (Nakamura 16)

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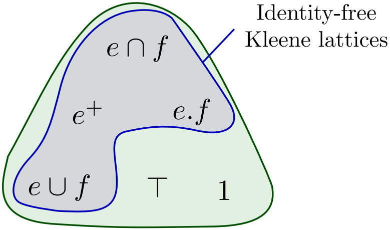


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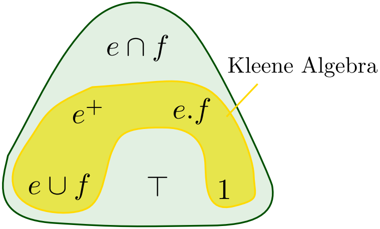


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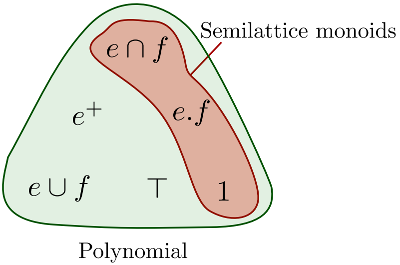

# Axiomatizing the equational theory of Relation Algebra

## Axiomatization

- **A set of axioms of the form**

$$e = f \qquad \text{or} \qquad e = f \Rightarrow g = h$$

- **Deduction rules**

$$e = f \ \wedge \ f = g \ \Rightarrow \ e = g \qquad \text{and} \qquad e = f \Rightarrow e\sigma = f\sigma$$

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Find a set of (quasi-)equations axiomatizing the equational theory of relations.

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- Solve hard instances by hand
- Gives certificates

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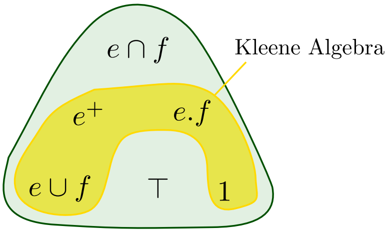


Quasi-axiomatizable (Kozen 94)

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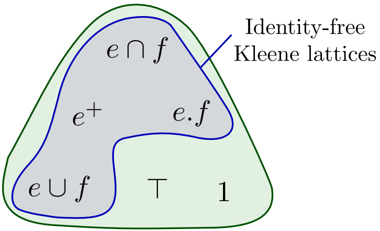


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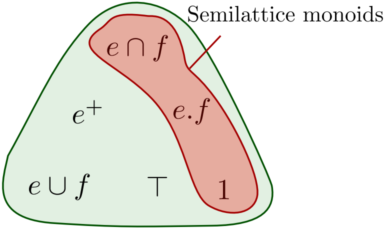


Non-axiomatizable (D. & Pous 20)

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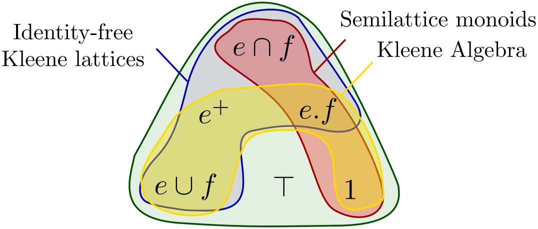
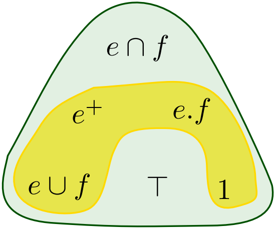


# Overview on Kleene Algebra



# KA expressions & languages

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

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## Language of a regular expression $\mathcal{L}(e)$

$$\mathcal{L}(a \cdot (b \cup c)) = \{ab, ac\} \qquad \mathcal{L}(a \cdot 1) = \{a\}$$

$$\mathcal{L}\left(a^+\right) = \{a, aa, \dots\}$$

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## Theorem (Pratt 1980)

$$\mathrm{Rel} \models e \subseteq f \ \Leftrightarrow \ \mathcal{L}(e) \subseteq \mathcal{L}(f)$$

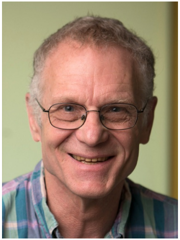
# Axiomatization

## Axioms of Kleene Algebra

- Axioms of an idempotent semiring decribing the behaviour of $\cup, \cdot, 1$.
- Two axioms describing the behaviour of $^+$:

$$f \cdot e \cup f \subseteq f \quad \Rightarrow \quad f \cdot e^+ \cup f \subseteq f$$
$$e \cup e \cdot e^+ \subseteq e^+$$

**We write** $\mathsf{KA} \vdash e \subseteq f$
**if** $e \subseteq f$ **follows from the axioms of Kleene Algebra.**



Dexter Kozen

## Theorem (Kozen 1994)

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad KA \vdash e \subseteq f$$

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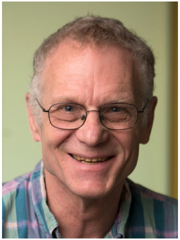

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## Theorem (Soundness)

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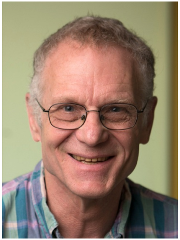

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## Theorem (Completeness)

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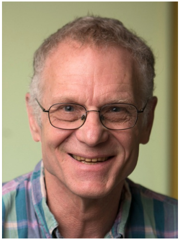
## Theorem (Completeness)

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad \mathcal{L}(e) \subseteq \mathcal{L}(f) \quad \Rightarrow \quad KA \vdash e \subseteq f$$



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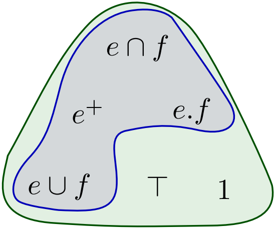
# Identity-free Kleene Lattices



# $\mathrm{KL}^-$ expressions & languages

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$$e, f \in ::= 1 \mid a \mid e \cdot f \mid \mathbf{e \cap f} \mid e \cup f \mid e^+$$

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## Language characterization

$$\mathrm{Rel} \models e \subseteq f \;\overset{?}{\Leftrightarrow}\; \mathcal{L}(e) \subseteq \mathcal{L}(f)$$

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# $\mathrm{KL}^{-}$ expressions & languages

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## Language characterization

$$\mathrm{Rel} \models e \subseteq f \;\not\Leftarrow\; \mathcal{L}(e) \subseteq \mathcal{L}(f)$$

$$\mathcal{L}(a \cap b) \subseteq \mathcal{L}(c) \qquad \text{but} \qquad \mathrm{Rel} \not\models a \cap b \subseteq c$$

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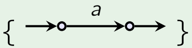
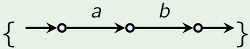
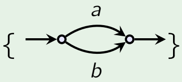
**Another notion of language is needed!**

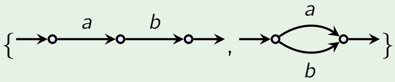
# Language of a KL$^-$ expression

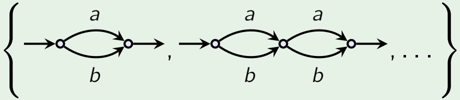
## Graph language of an expression $\mathcal{G}(e)$

$\mathcal{G}(a) =$ 


$\mathcal{G}(a \cdot b) =$ 


$\mathcal{G}(a \cap b) =$ 


$\mathcal{G}(a.b \cup a \cap b) =$ 


$\mathcal{G}\left((a \cap b)^+\right) =$ 

# Characterization theorem

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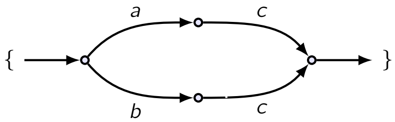
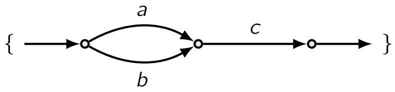



$(a \cdot c) \cap (b \cdot c)$

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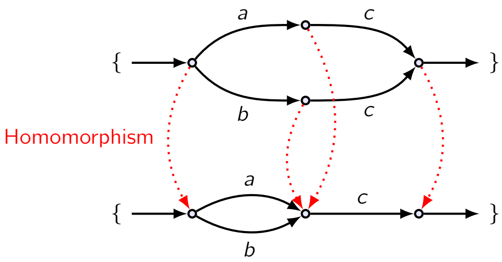


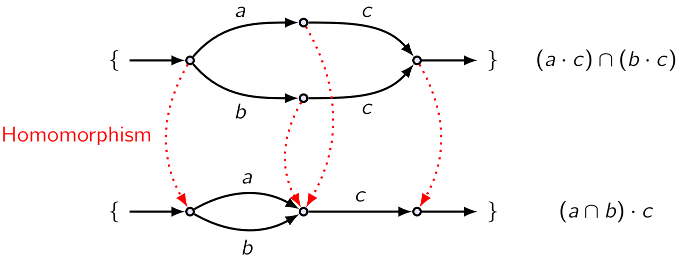

$(a \cdot c) \cap (b \cdot c)$

$(a \cap b) \cdot c$

# Characterization theorem

$$\mathrm{Rel} \models e \subseteq f \;\not\Rightarrow\; \mathcal{G}(e) \subseteq \mathcal{G}(f)$$

$$\mathrm{Rel} \models (a \cap b) \cdot c \quad \subseteq \quad (a \cdot c) \cap (b \cdot c)$$



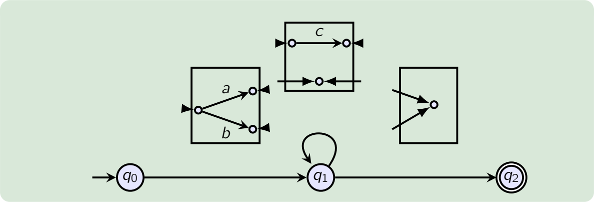

Paul Brunet

Damien Pous

**Theorem [Brunet & Pous, LICS 2015]**

$$\mathrm{Rel} \models e \subseteq f \qquad \Leftrightarrow \qquad \mathcal{G}(e) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(f)$$

# Graph automata



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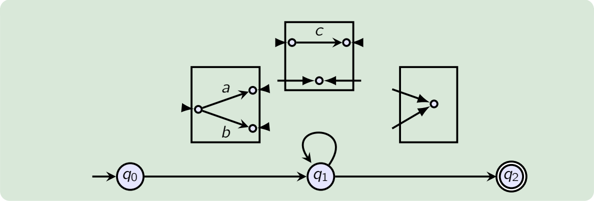
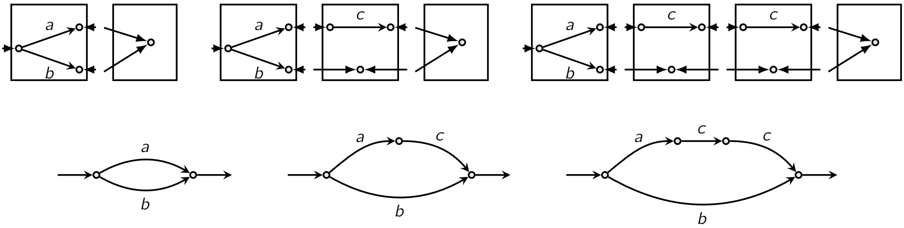



**Runs:**



# Kleene theorem & Decidability

## Theorem [Brunet & Pous LICS 2015]

For every graph automaton $P$, there is an expression $e$ such that

$$\mathcal{G}(e) = \mathcal{G}(P)$$

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### Theorem [Brunet & Pous LICS 2015]

For every graph automata $P, Q$, the property $\mathcal{G}(P) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(Q)$ is decidable.

# Axiomatization

## Axioms of Kleene lattices

- Axioms of Kleene algebra.
- Axioms os a distributive lattice describing the behavior of $\cup, \cap$.

**We write** $\mathsf{KL}^- \vdash e \subseteq f$
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## Correction

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$$\mathcal{G}(e) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(f) \quad \Rightarrow \quad \mathsf{KL}^- \ \vdash \ e \subseteq f$$

## Weak completeness

$$\mathcal{G}(e) \subseteq \mathcal{G}(f) \quad \Rightarrow \quad \mathsf{KL}^- \ \vdash \ e \subseteq f$$

# Synchronized Kleene theorem

## Theorem

If $P$ and $Q$ are graph automata such that $\mathcal{G}(P) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(Q)$, then there are two expressions $e$ and $f$ such that:

$$\mathcal{G}(e) = \mathcal{G}(P), \quad \mathcal{G}(f) = \mathcal{G}(Q) \quad \text{and} \quad \mathsf{KL}^{-} \vdash e \subseteq f.$$

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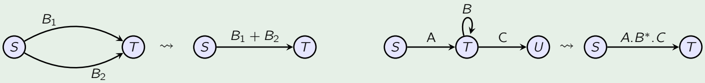
Proof:

**State elimination:**



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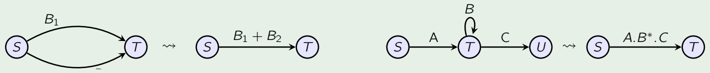
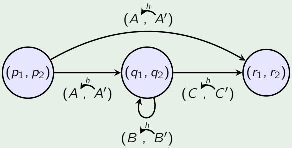



**Product automaton:**



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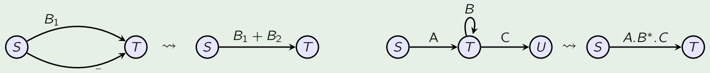


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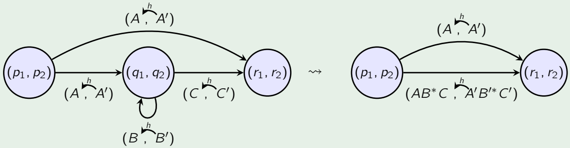

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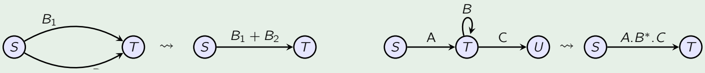


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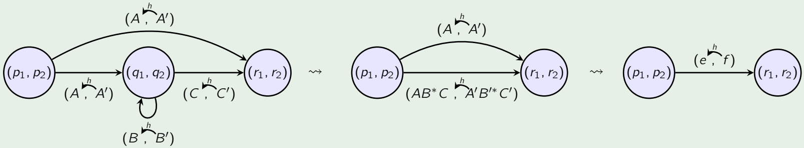

# Completeness proof

## Theorem

$$\mathrm{Rel} \models e \subseteq f \quad \Rightarrow \quad \mathsf{KL}^- \vdash e \subseteq f$$

Proof:

$e$ $\qquad$ $\mathrm{Rel} \models e \subseteq f$ $\qquad$ $f$

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$$\begin{tikzcd}
e \arrow[dr, "\mathcal{G}(e) = \mathcal{G}(P)"'] & & \mathcal{G}(e) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(f) & & f \arrow[dl, "\mathcal{G}(Q) = \mathcal{G}(f)"] \\
& P & \mathcal{G}(P) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(Q) & Q &
\end{tikzcd}$$

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e' & & {\color{red}\mathsf{KL}^- \ \vdash \ e' \subseteq f'} & & f'
\end{tikzcd}$$

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$\mathcal{G}(e) \subseteq \mathcal{G}(e')$

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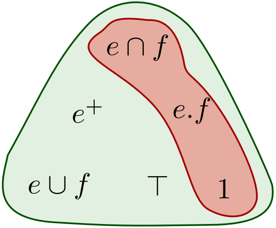
$\color{red}{\mathsf{KL}^- \ \vdash\ f' \subseteq f}$

# Semilattice monoids



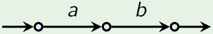
# SLM expressions & languages

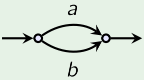
Let $\Sigma = \{a, b, \ldots\}$ be a finite alphabet.

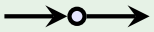
## SLM expressions

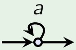
$$e, f \in ::= a \mid e \cdot f \mid e \cap f \mid 1$$

## Graph of an expression $\mathcal{G}(e)$

$\mathcal{G}(a \cdot b) =$ 


$\mathcal{G}(a \cap b) =$ 


$\mathcal{G}(1) =$ 

$\mathcal{G}(a \cap 1) =$ 


## Characterization theorem [Freyd & Scedrov 90]

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad \mathcal{G}(e) \blacktriangleleft \mathcal{G}(f)$$

# Decidability & Non-axiomatizability

**Theorem**

The equational theorey is decidable for SLM expressions.

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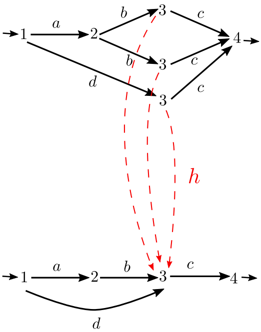
**Theorem [D. & Pous 2020]**

The equational theory is not axiomatizable for SLM expressions.

# Degrees and $n$-decompositions of homomorphisms




deg($h$)=3

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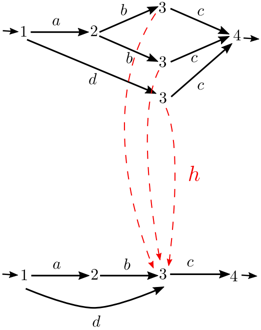
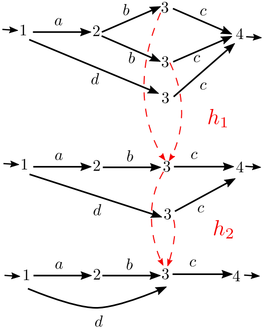



$\deg(h){=}3$




$\deg(h_1){=}2$ $\deg(h_1){=}2$

# Homomorphism decomposition

**Proposition [D. & Pous 2019]**

The equational theory of SLM is axiomatizable

$\Downarrow$

$\exists n$ every homomorphism of SLM expressions is $n$-decomposable.

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The equational theory of SLM is axiomatizable

$\Downarrow$

$\exists n$ every homomorphism of SLM expressions is $n$-decomposable.

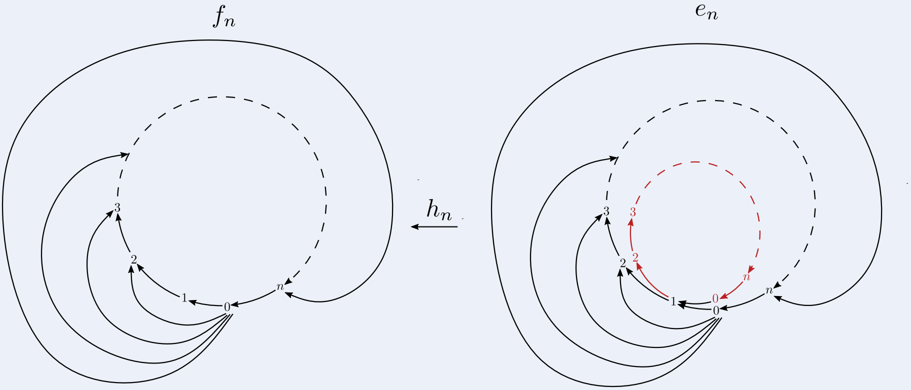
Find $(e_n, f_n)_{n \in \omega}$ SLM expressions such that:

- $h_n : e_n \to f_n$,
- $h_n$ is not $m$-decomposable for every $m < n$.

# The counter-example

## Theorem (D. & Pous 2019)

For every $n$, the following homomorphism




is not $m$-decomposable for every $m < n$.

# Future work

- Find a general framework for decidability and axiomatizability proofs.
- What about non-quasi-axiomatizability?
- Sufficient conditions for non-axiomatizability.

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**Thank you for your attention !**