Characterizing Double Categories of Relations

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Axiomatization of Relations

A **relation** in a category is a monomorphism $R \to A \times B$.

$\mathscr{C}$ = regular category

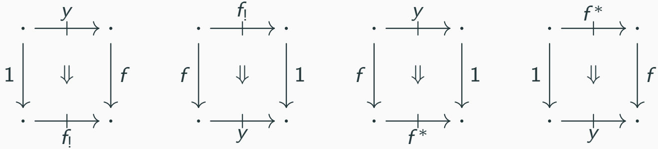
1. $\mathscr{C} \rightsquigarrow \mathrm{Rel}(\mathscr{C})$ ordinary category/locally discrete 2-category
2. $\mathscr{C} \rightsquigarrow \mathfrak{Rel}(\mathscr{C})$ a bicategory
3. $\mathscr{C} \rightsquigarrow \mathbb{R}\mathrm{el}(\mathscr{C})$ a double category.

Question

- An **allegory** [FS90] is a locally-ordered 2-category equipped with an anti-involution $(-)^\circ$ satisfying a modularity law; every tabular allegory is equivalent to a category $\mathsf{Rel}(\mathscr{C})$.
- A **bicategory of relations** [CW87] is a cartesian bicategory in which every object is discrete. Every functionally complete bicategory of relations is biequivalent to one of the form $\mathfrak{Rel}(\mathscr{C})$.
- Which double categories $\mathbb{D}$ are equivalent to those of the form $\mathbb{R}\mathsf{el}(\mathscr{C})$?

Double Categories

- A **double category** is a pseudo-category object in Cat.
- A double category $\mathbb{D}$ is an **equipment** [Shu08] if every arrow f has a **companion** and a **conjoint**: proarrows $f_!$ and f^* and cells



  satisfying some equations [GP04].
- $\mathbb{D}$ has **tabulators** if $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ has a right adjoint.
- [Ale18] A double category $\mathbb{D}$ is **cartesian** if $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $\mathbb{D} \to 1$ have right adjoints in Dbl.
- Cartesian equipments $\mathbb{S}\mathrm{pan}(\mathscr{C})$ characterized in [Ale18]
- Goal: characterize cartesian equipments of the form $\mathbb{R}\mathrm{el}(\mathscr{C})$.

Spans

An **oplax/lax adjunction** is a conjoint pair in the strict double category of double categories with oplax and lax functors.

Theorem (§5 of [Nie12])
Let $\mathbb{D}$ denote a double category with pullbacks. The following are equivalent:

1. *The identity functor* $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ *extends to an oplax/lax adjunction* $F\colon \mathbb{S}\mathrm{pan}(\mathbb{D}_0) \rightleftarrows \mathbb{D}\colon G$*.*
2. $\mathbb{D}$ *is an equipment with tabulators.*

Spans

The oplax/lax adjunction is a strong (both functors are pseudo!) equivalence under some further completeness conditions:

Theorem (§5 of [Ale18])
For a double category $\mathbb{D}$ the following are equivalent:

1. $\mathbb{D}$ *is equivalent to* $\mathbb{S}\mathrm{pan}(\mathscr{C})$ *for finitely-complete* $\mathscr{C}$*.*
2. $\mathbb{D}$ *is a unit-pure cartesian equipment admitting certain Eilenberg-Moore objects.*
3. $\mathbb{D}_0$ *has pullbacks satisfying a Beck-Chevalley condition and the canonical functor* $\mathbb{S}\mathrm{pan}(\mathbb{D}_0) \to \mathbb{D}$ *is an equivalence of double categories.*

Relations as a Double Category

- Let $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ be a proper and stable factorization system on a finitely complete category $\mathscr{C}$.
- The double category $\mathbb{R}\mathsf{el}(\mathscr{C}; \mathcal{F})$ is a cartesian equipment.
- "Local products" [Ale18] in $\mathbb{R}\mathsf{el}(C; \mathcal{F})$ satisfy a Frobenius law:

$$R \wedge f^* \otimes S \cong f^* \otimes (f_! \otimes R \wedge S)$$

- $\mathbb{R}\mathsf{el}(\mathscr{C}; \mathcal{F})$ is "unit-pure" i.e. $y \colon \mathbb{D}_0 \to \mathbb{D}_1$ is fully faithful.
- $\mathbb{R}\mathsf{el}(\mathscr{C}; \mathcal{F})$ has tabulators.
- Eilenberg-Moore implies tabulators for spans. But tabulators are basic for allegories and bicategories of relations.

Tabulators

- An allegory is **tabular** if every arrow R has a tabulator: a pair of arrows f and g such that
    1. $gf^\circ = R$ "tabulators are strong"
    2. $f^\circ f \wedge g^\circ g = 1$ "tabulators are monic."
- A double category $\mathbb{D}$ has **tabulators** if $y\colon \mathbb{D}_0 \to \mathbb{D}_1$ has a right adjoint $\top\colon \mathbb{D}_1 \to \mathbb{D}_0$ in Dbl. The **tabulator** of $m\colon A \nrightarrow B$ is the object $\top m$ together with a counit cell $\top m \Rightarrow m$.
- tabulators in $\mathbb{R}\mathrm{el}(\mathscr{C}; \mathcal{F})$ satisfy:
    1. $\langle l, r\rangle\colon \top m \to A \times B$ is in $\mathcal{M}$ and $l^* \otimes l_! \wedge r^* \otimes r_! \cong y$ holds (tabulators are monic)
    2. and $m \cong l^* \otimes r_!$ holds (strong).

Niefield's Theorem for Relations

Theorem
For a double category $\mathbb{D}$ with a stable and proper factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ on $\mathbb{D}_0$, the identity functor $\mathbb{D}_0 \to \mathbb{D}_0$ extends to a normalized oplax/lax adjunction $F\colon \mathbb{R}\mathrm{el}(\mathbb{D}_0; \mathcal{F}) \rightleftarrows \mathbb{D}\colon G$ if, and only if,

1. *$\mathbb{D}$ is a unit-pure equipment;*
2. *has $\mathcal{M}$-monic tabulators;*
3. *the unit cell y_e is an extension for each $e \in \mathcal{E}$.*

A First Pass on the Characterization

Theorem
The identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ *extends to an adjoint equivalence of pseudo-functors*

$$F\colon \mathbb{R}\mathsf{el}(\mathbb{D}_0; \mathcal{F}) \rightleftarrows \mathbb{D}\colon G$$

if, and only if,

1. $\mathbb{D}$ *is a unit-pure equipment;*
2. y_e *is an extension cell for each cover* e*;*
3. $\mathbb{D}$ *has strong,* $\mathcal{M}$*-monic and externally functorial tabulators;*
4. *every* $\mathcal{F}$*-relation* $R \to A \times B$ *is a tabulator of its canonical extension.*

Covers and Inclusions

Definition (Cf. [Sch15])
The **kernel** of a morphism $f\colon A \to B$ is the restriction ρ of the unit on B along f. Dually, the **cokernel** of f is the extension cell ξ

$$
\begin{array}{ccc}
A & \xrightarrow{f_! \otimes f^*} & A \\
{\scriptstyle f}\downarrow & \rho & \downarrow{\scriptstyle f} \\
B & \xrightarrow[y_B]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{y_A} & A \\
{\scriptstyle f}\downarrow & \xi & \downarrow{\scriptstyle f} \\
B & \xrightarrow[f^* \otimes f_!]{} & B
\end{array}
$$

A morphism $e\colon A \to E$ in an equipment is a **cover** if the canonical globular cell is an iso $e^* \otimes e_! \cong y_E$. Dually, a morphims $m\colon E \to B$ is an **inclusion** if the canonical globular cell is an iso $m_! \otimes m^* \cong y_E$.

A Factorization System

Theorem
Suppose that $\mathbb{D}$ is a unit-pure cartesian equipment with strong and $\mathcal{M}$-monic tabulators. If local products satisfy Frobenius and every relation is a tabulator, then with $\mathcal{E}$ = covers and $\mathcal{M}$ = inclusions, $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ is a proper, stable factorization system on $\mathbb{D}_0$.

Given $f\colon A \to B$, a factorization arises as in the diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\;y\;} & B \\
{\scriptstyle e}\downarrow & \exists\,! & \downarrow{\scriptstyle e} \\
\top(f^* \otimes f_!) & \xrightarrow{\;y\;} & \top(f^* \otimes f_!) \\
{\scriptstyle l}\downarrow & \tau & \downarrow{\scriptstyle r} \\
B & \xrightarrow[f^* \otimes f_!]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{\;y\;} & B \\
{\scriptstyle f}\downarrow & \xi & \downarrow{\scriptstyle f} \\
B & \xrightarrow[f^* \otimes f_!]{} & B
\end{array}
$$

using the universal property of the tabulator.

Theorem

If $\mathbb{D}$ is a unit-pure, cartesian equipment such that

1. *tabulators exist and are strong, $\mathcal{M}$-monic and functorial,*
2. *every relation is a tabulator, and finally*
3. *local products satsify Frobenius*

then the identity functor $1\colon \mathbb{D}_0 \to \mathbb{D}_0$ extends to an adjoint equivalence

$$\mathbb{R}\mathsf{el}(\mathbb{D}_0; \mathcal{F}) \simeq \mathbb{D}$$

where $\mathcal{F}$ is the orthogonal factorization system given by inclusions and covers.

Theorem
A double category $\mathbb{D}$ *is equivalent to one* $\mathbb{R}\mathsf{el}(\mathscr{C};\mathcal{F})$ *for some proper and stable factorization system on a finitely-complete category* $\mathscr{C}$ *if, and only if,* $\mathbb{D}$ *is*

1. *a unit-pure cartesian equipment;*
2. *with strong,* $\mathcal{M}$*-monic and functorial tabulators;*
3. *in which every relation is the tabulator of its cokernel;*
4. *and for which local products satisfy Frobenius.*

For more see [Lam21].

THANK YOU!

Evangelia Aleiferi.
Cartesian Double Categories with an Emphasis on Characterizing Spans.
arXiv e-prints, page arXiv:1809.06940, September 2018.

A. Carboni and R.F.C. Walters.
Cartesian bicategories I.
Journal of Pure and Applied Algebra, 49:11–32, 1987.

Peter J. Freyd and Andre Scedrov.
Categories, Allegories.
North Holland, 1990.

Marco Grandis and Robert Paré.
Adjoints for double categories.
Cahiers de Topologie et Géom. Diff. Catégoriques, 45:193–240, 2004.

Michael Lambert.
Double Categories of Relations.
arXiv e-prints, page arXiv:2107.07621, July 2021.

Susan Niefield.
Span, cospan, and other double categories.
Theory and Applications of Categories, 26(26):729–742, 2012.

Patrick Schultz.
Regular and exact (virtual) double categories.
arXiv e-prints, page arXiv:1505.00712, May 2015.

Michael Shulman.
Framed bicategories and monoidal fibrations.
Theory and Applications of Categories, 20:650–738, 2008.