Towards foundations of categorical cybernetics

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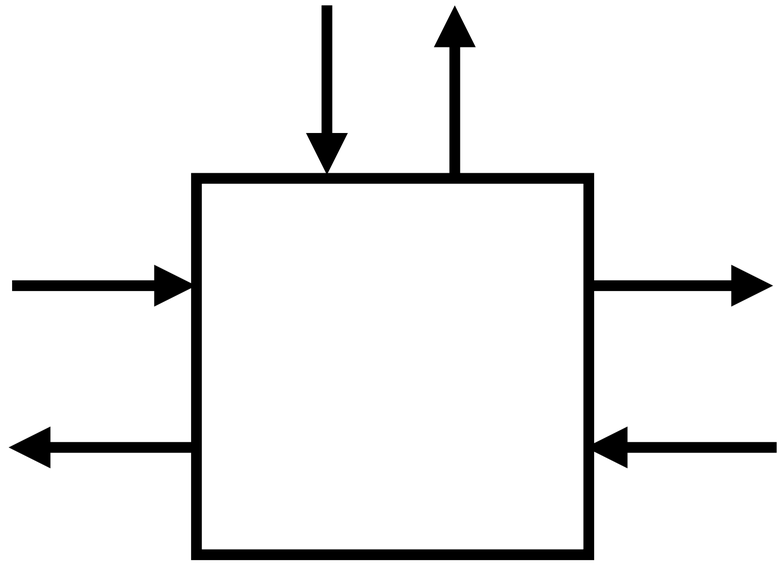
The goal

A monoidal category of processes that:

- Depend on an external parameter
- Propagate back “responses” to the environment
- Also propagate back “responses” to the parameter’s controller



The Para construction

Let $\mathscr{C}$ be a symmetric monoidal category

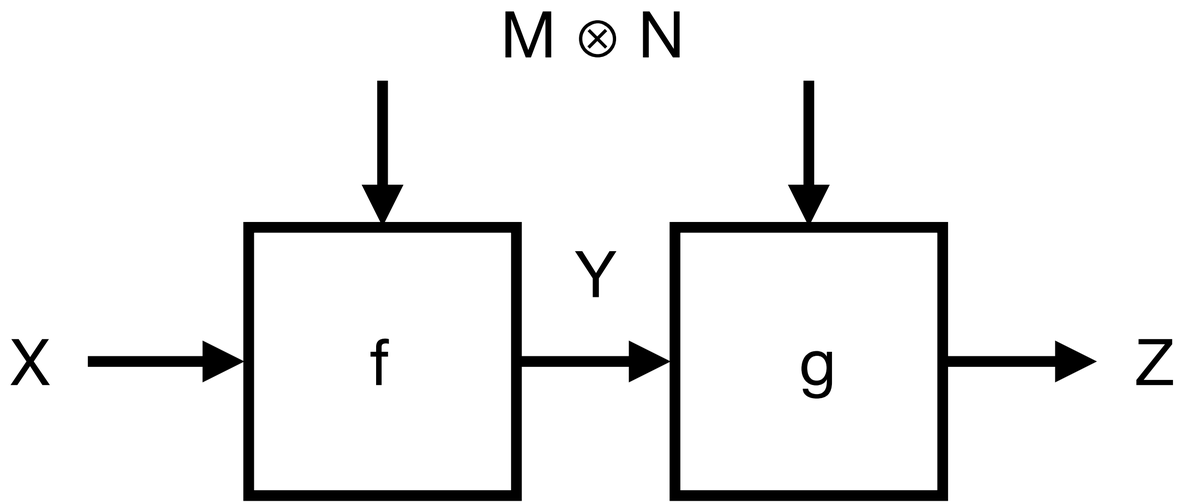
Para($\mathscr{C}$) is a symmetric monoidal bicategory where:

- Objects = objects of $\mathscr{C}$
- Morphisms $X \to Y$ = pairs ($M \in \mathscr{C}$, $f : M \otimes X \to Y$)
- 2-cells $(M, f) \to (N, g)$ = “reperameterisations” $h : M \to N$

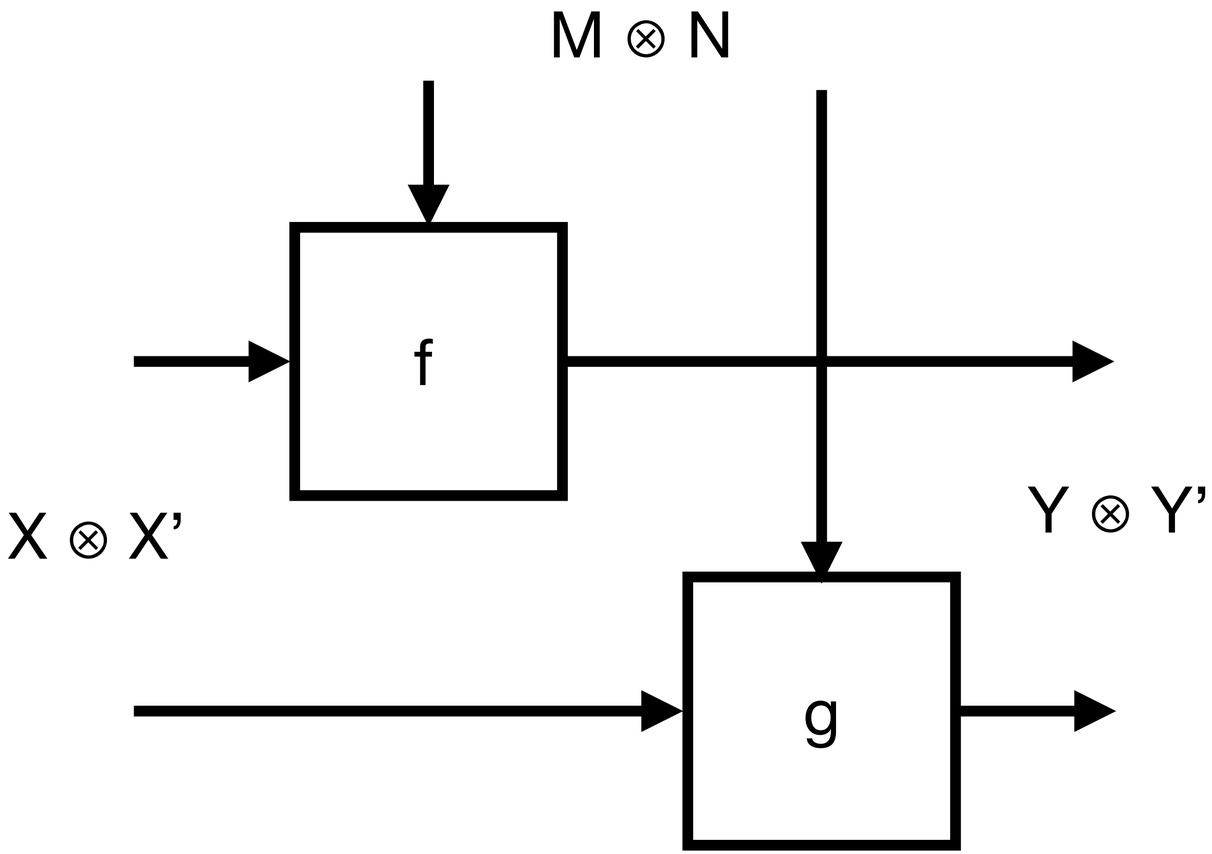
(Nb. This generalises from monoidal categories to actegories)

Picturing morphisms of Para

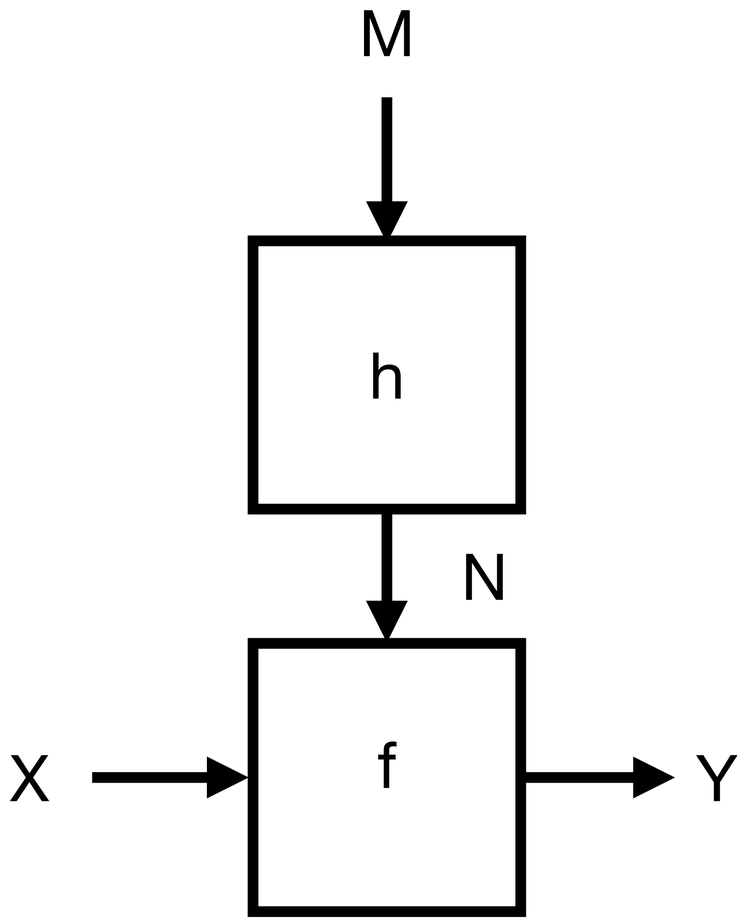
Sequential composition



Parallel composition

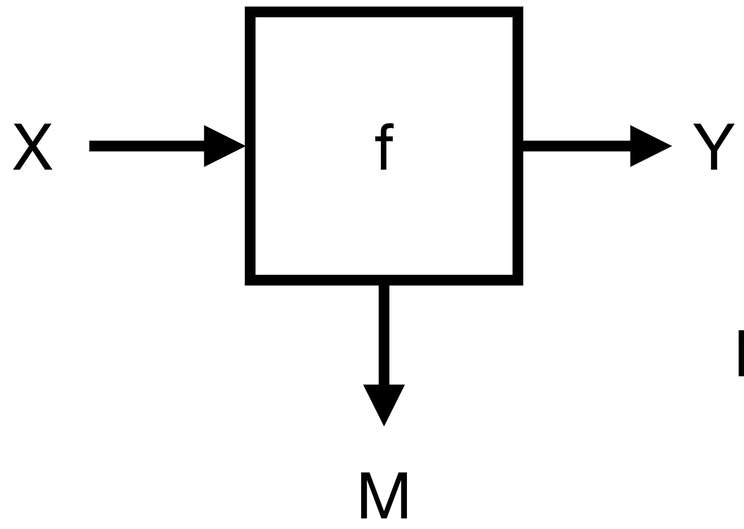


Reperameterisation



hom-categories of Para($\mathscr{C}$) are fibred over $\mathscr{C}$

CoPara



Morphisms $X \to Y$ = pairs $(M \in C, f : X \to M \otimes Y)$

$\mathrm{CoPara}(\mathcal{C})^{\mathrm{op}} = \mathrm{Para}(\mathcal{C}^{\mathrm{op}})$

Optics = CoPara + Para

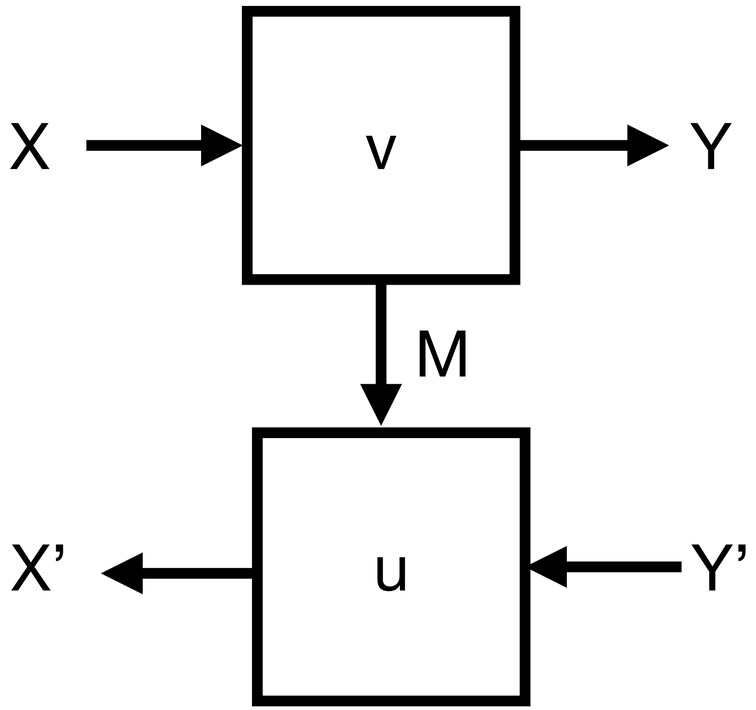
Optic($\mathcal{C}$) is a monoidal category where:

- Objects = pairs of objects of C
- Morphisms (X, X') → (Y, Y') = equivalence classes of triples

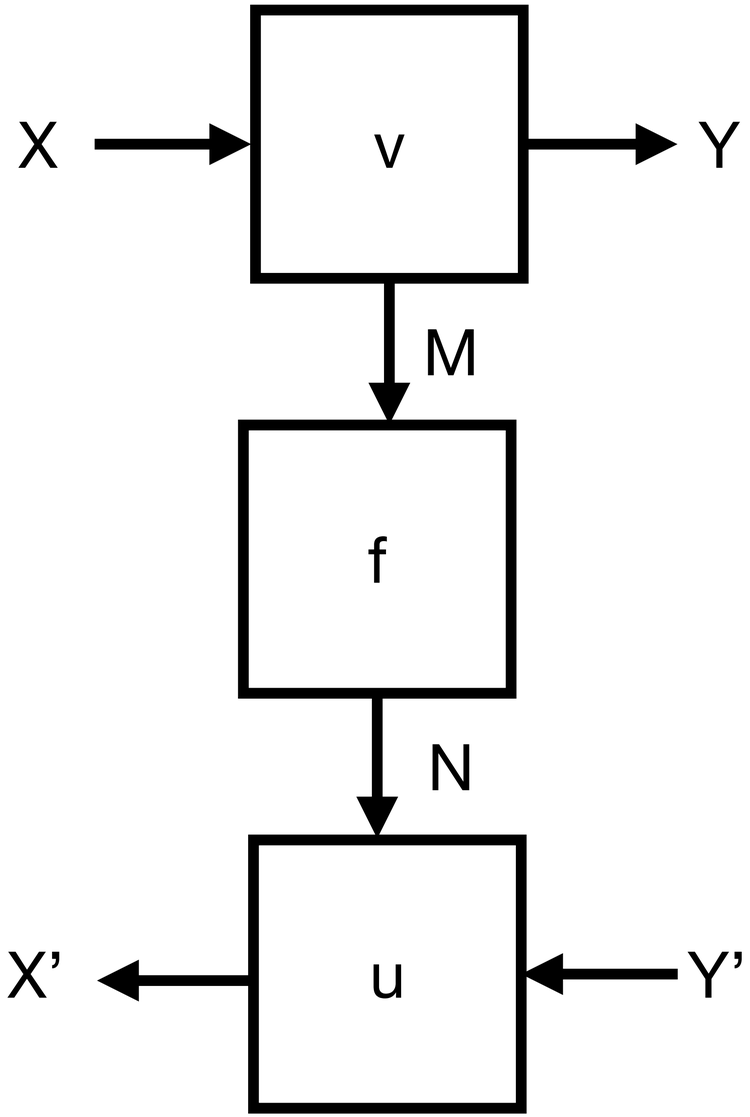
  $M \in \mathcal{C}$

  $v : X \to M \otimes Y$

  $u : M \otimes Y' \to X'$



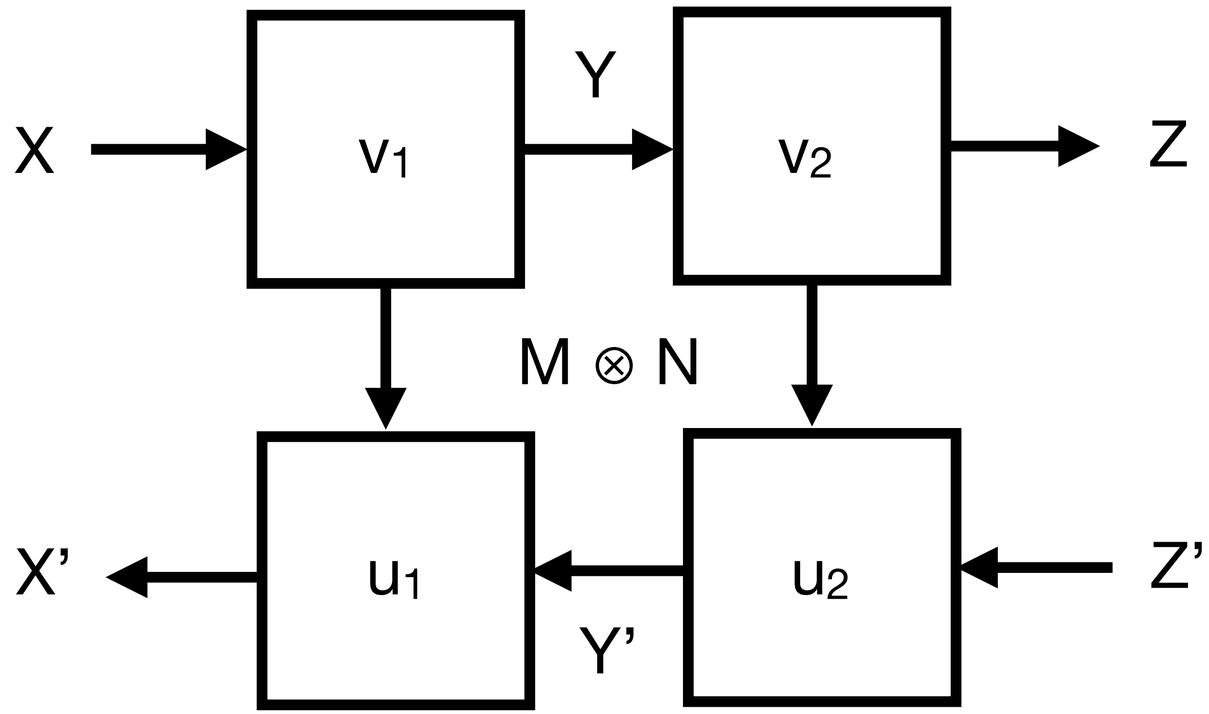
Equivalence of optics



$(M, v, f^{\star}u) \sim (N, f_{\star}v, u)$

(It’s a coend)

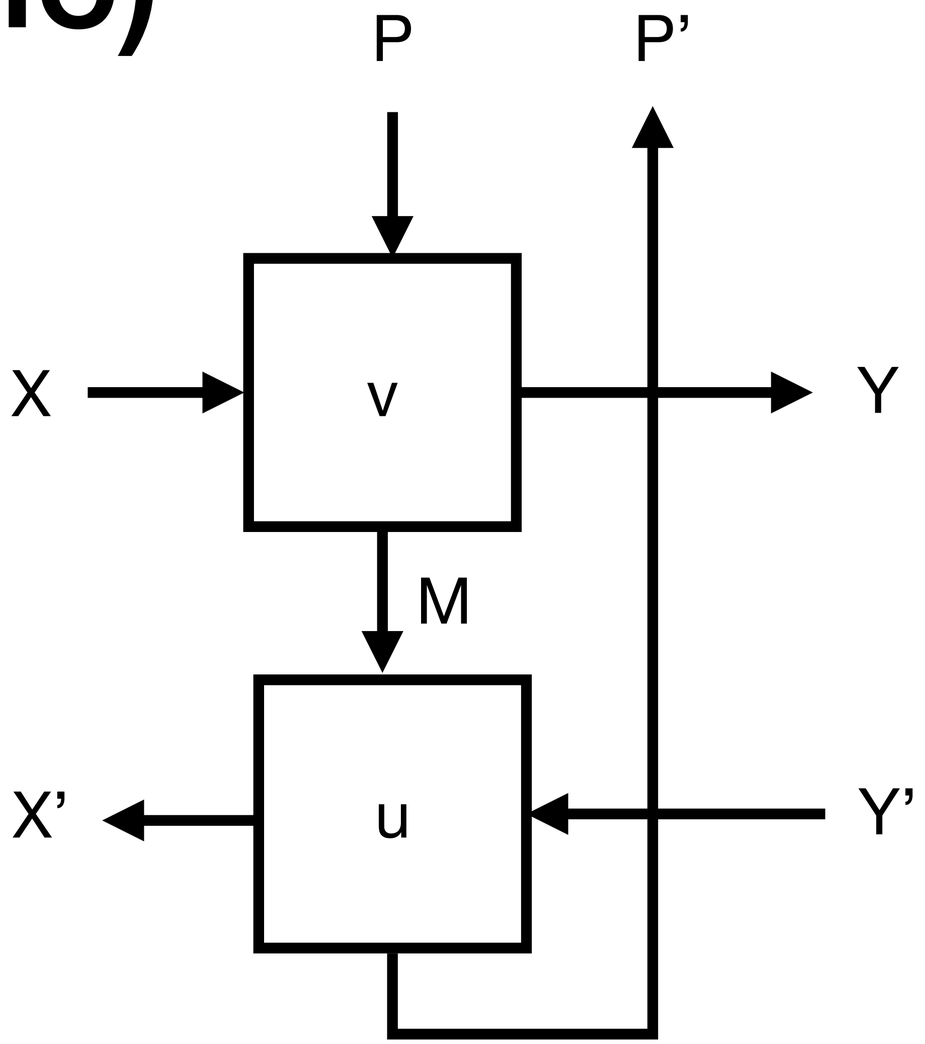
Optic composition



Para(Optic)

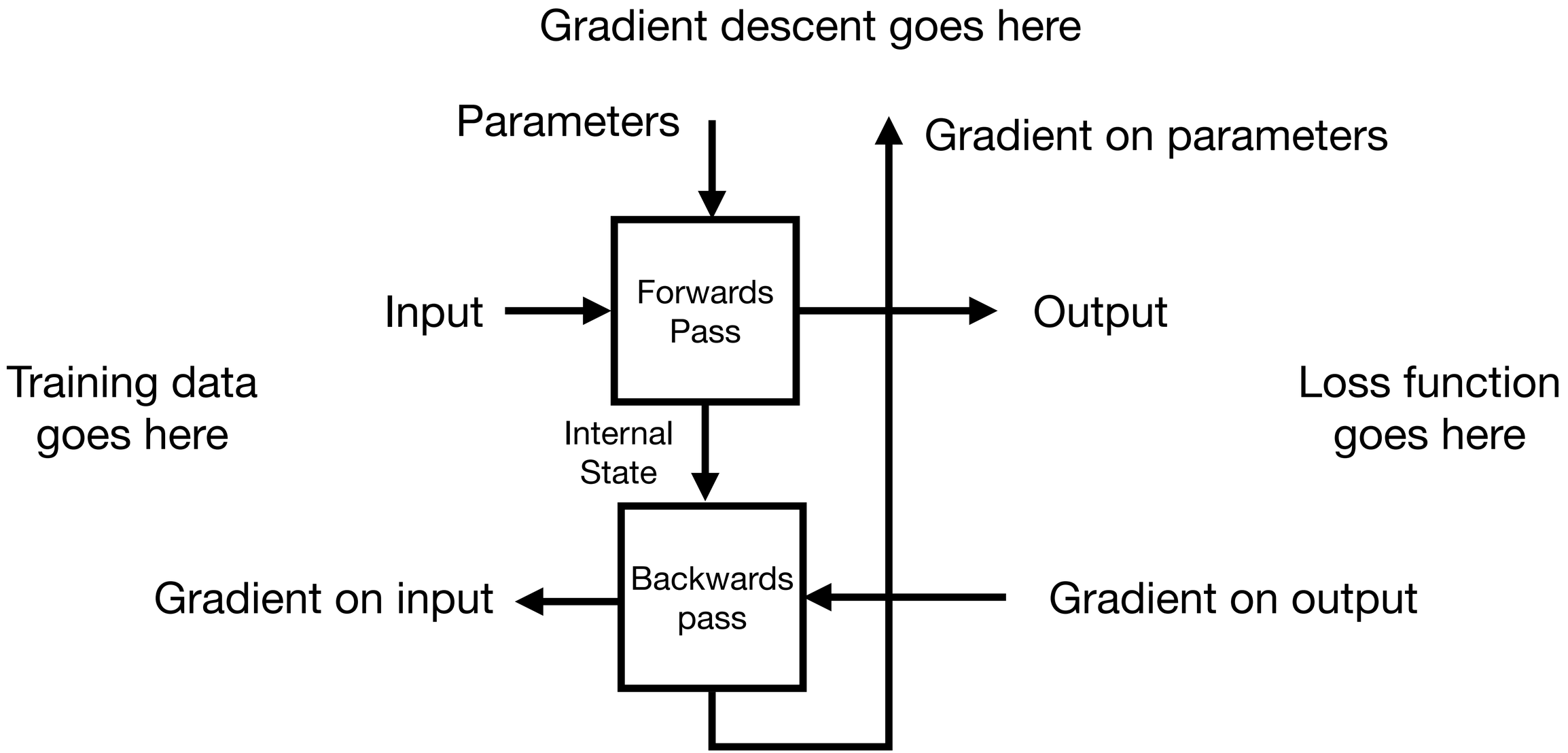
Para(Optic($\mathcal{O}$)) is a monoidal bicategory with:

- Objects = pairs of objects of C
- Morphisms $(X, X') \to (Y, Y')$ =
  $P \in C, P' \in C, (P, P') \otimes (X, X') \to (Y, Y')$

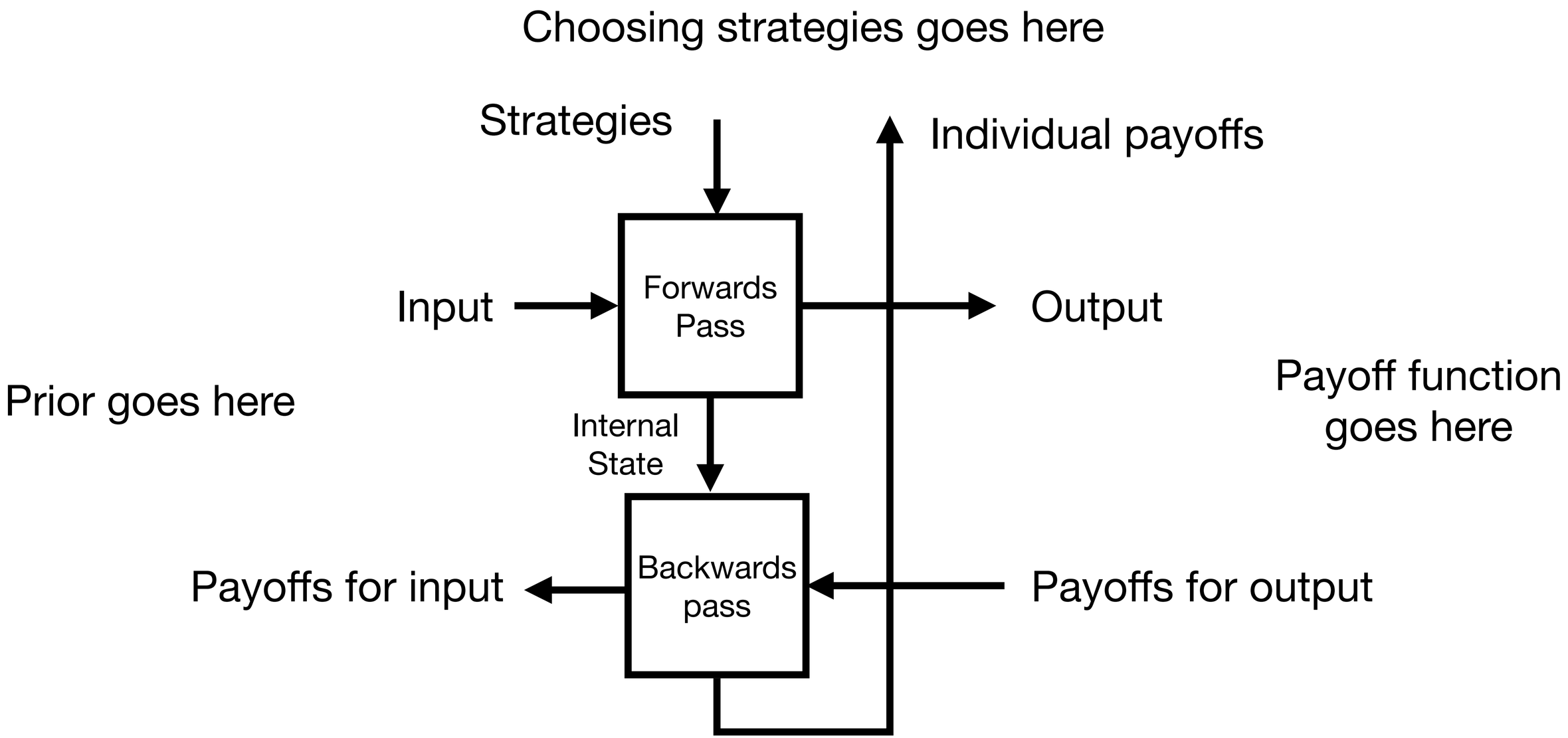


Central claim: Para(Optic) is the right setting for “cybernetic” processes

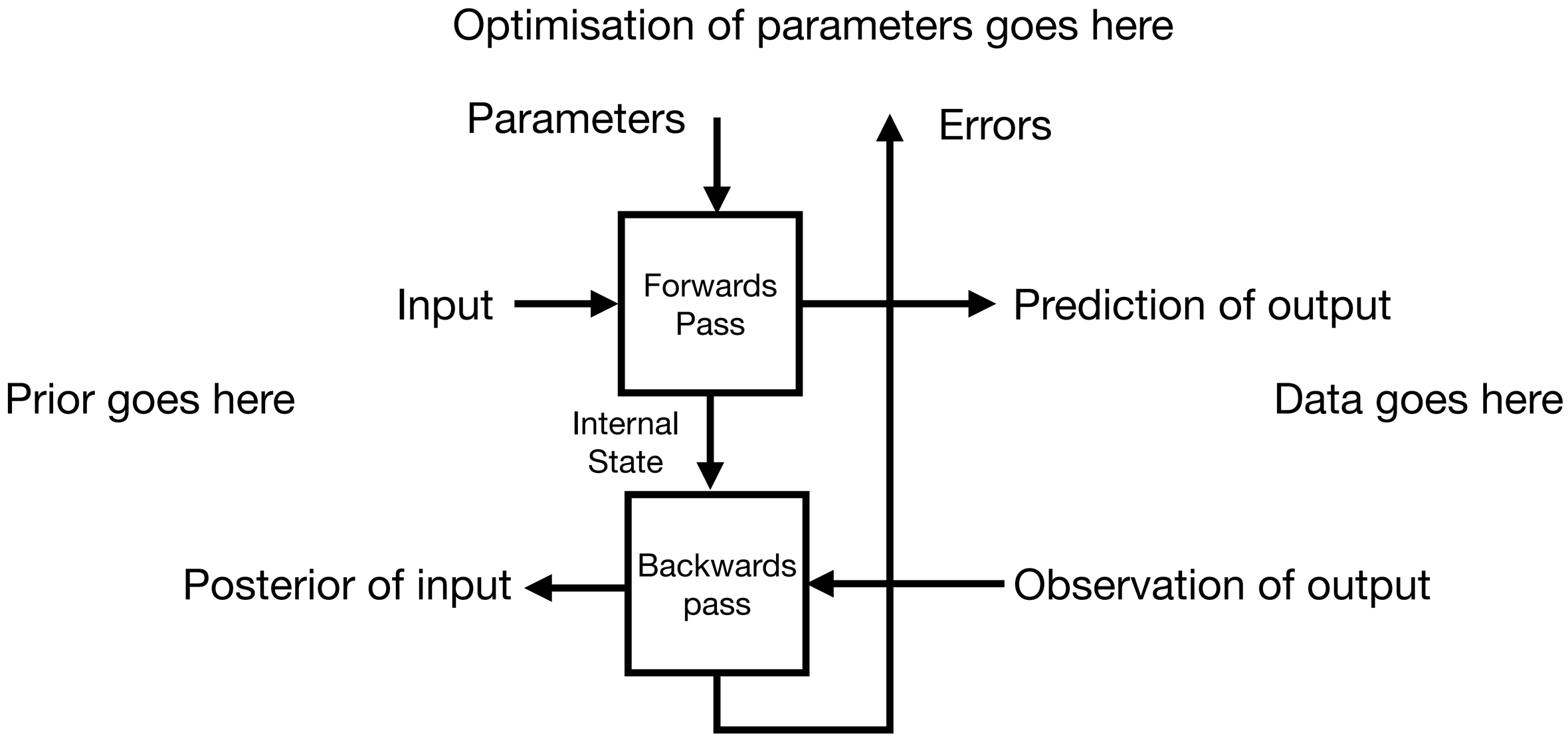
Example: Supervised learning



Example: Game theory



Example: Variational inference



Example: Reinforcement learning