Profunctors between posets and preserving cuts

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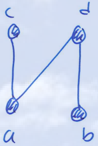
Overview

1. Posets and profunctors
2. Examples
3. Main statement
4. Applications

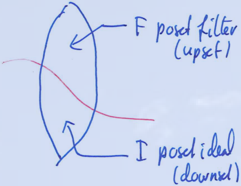
Posets and cuts

- Hasse diagram:

Ex



- Cut:



$\rightsquigarrow (I, F)$

- Distributive lattice $\hat{P}$: All cuts (I, F) ordered by:

$(I, F) \leq (J, G)$ if $I \subseteq J$.

Distributive lattices

- Order-preserving maps: $f : P \to Q$ ⤳ $\mathrm{Hom}(P, Q)$
- $\hat{P} = \mathrm{Hom}(P^{op}, \{0 < 1\})$, P^{op} is opposite poset
- $\hat{P}$ has top element $(P, \emptyset) = \infty$ and bottom element $(\emptyset, P) = \emptyset$
- Examples
  - A set, $\hat{A}$ = subsets of A = Boolean poset on A
  - $P = [n] = \{1 < 2 < \cdots < n\}$, $\hat{[n]} = [n+1] = [n] \cup \{\infty\}$ where $\infty = n+1$.

Profunctors

- **Profunctor** $P \rightarrowtail Q$ is order-preserving map $P \to \hat{Q}$.

- $\mathrm{Hom}_{pro}(P, Q) = \mathrm{Hom}(P, \hat{Q}) = \mathrm{Hom}(P, \mathrm{Hom}(Q^{op}, \{0<1\}))$

$$= \mathrm{Hom}(P \times Q^{op}, \{0<1\})$$

$$= \mathrm{Hom}((P^{op} \times Q)^{op}, \{0<1\})$$

- **Profunctor** $P \rightarrowtail Q \overset{1-1}{\longleftrightarrow}$ cuts in $P^{op} \times Q$

Profunctors for sets and categories

- A, B <u>sets</u>: $A \nrightarrow B \overset{1-1}{\longleftrightarrow}$ cut in $A^{op} \times B$

  $\overset{1-1}{\longleftrightarrow}$ subset of $A^{op} \times B$

  $\overset{1-1}{\longleftrightarrow}$ relation between A and B

- C, D <u>categories</u>: $C \nrightarrow D$ profunctor

  is a functor $C \longrightarrow \mathrm{Hom}(D^{op}, \underline{Set})$

  ↑ presheaves on D.

Example

$$f : [6] \nrightarrow [4].$$

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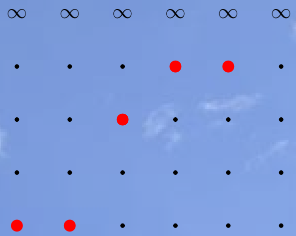
- Same as cut in:

$$[6]^{op} \times [4].$$

Graph

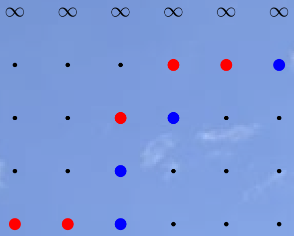
Profunctor $f : [6] \nrightarrow [4]$



Red circles: Graph Γf

Ascent

Profunctor $f : [6] \nrightarrow [4]$



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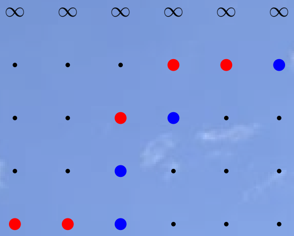

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Graph $\cup$ ascent is *boundary* of cut in $[6]^{op} \times [4]$

Profunctors $f : P \nrightarrow Q$

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- Subsets of $UP^{\text{op}} \times UQ$.
- $\Gamma f \cup \Lambda f$ is the boundary of the cut of $P^{\text{op}} \times Q$ defined by f.

Graph and ascent maps

$$\Lambda : U\mathrm{Hom}_{pro}(P, Q) \to (UP^{\mathrm{op}} \times UQ)\widehat{}, \quad f \mapsto (\Lambda f, -)$$

$$\Gamma : U\mathrm{Hom}_{pro}(P, Q) \to (UP^{\mathrm{op}} \times UQ)\widehat{}, \quad f \mapsto (-, \Gamma f).$$

Extending to cuts

$(\mathcal{I}, \mathcal{F})$ cut for $\text{Hom}_{pro}(P, Q)$.

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$$\Gamma^{!} : \mathrm{Hom}_{pro}(P, Q)\widehat{\ } \to (UP^{op} \times UQ)\widehat{\widehat{\ }}, \quad (\mathcal{I}, \mathcal{F}) \mapsto (\Gamma(\mathcal{I})^{\downarrow}, -).$$

$\Gamma(\mathcal{I})^{\downarrow}$ is poset ideal generated by the cuts $(-, \Gamma f)$ for $f \in \mathcal{I}$.

P and Q are well-founded posets

Theorem (Preserving the cut)

Let $(\mathcal{I}, \mathcal{F})$ a cut for $Hom_{pro}(P, Q)$. Then $(\Gamma(\mathcal{I})^{\downarrow}, \Lambda(\mathcal{F})^{\uparrow})$ is a cut for $(UP^{op} \times UQ)\widehat{}$.

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$$(UP^{op} \times UQ)\widehat{\ } \cong \mathrm{Hom}_{pro}(UP, UQ).$$

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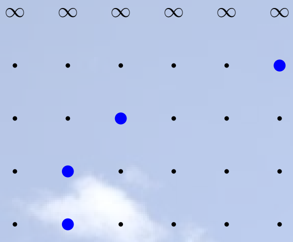
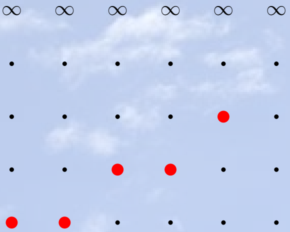


Ascents of $f \in \mathcal{F}$



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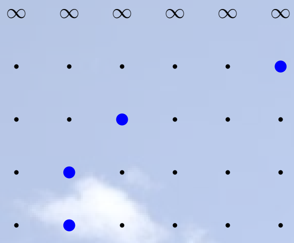

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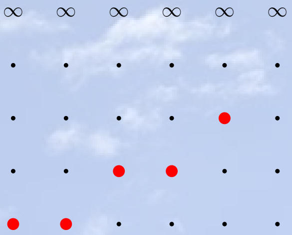

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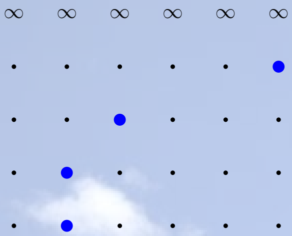

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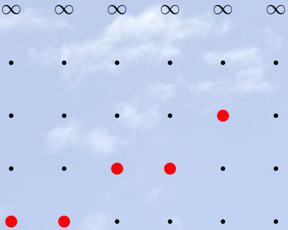

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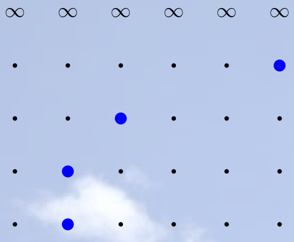

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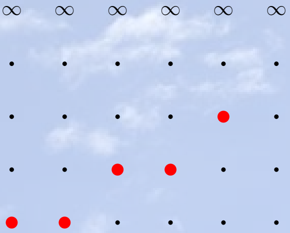

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- The complement S^c contains a red path "from $\mathcal{I}$".
- These two cases are mutually exclusive.

Open problem

Natural inclusions $UP \to P$ and $UQ \to Q$.

Can you get functorially from:

Cut $(\mathcal{I}, \mathcal{F})$ for $\text{Hom}_{pro}(P, Q)$, to

Cut $(\Gamma(\mathcal{I})^{\downarrow}, \Lambda(\mathcal{F})^{\uparrow})$ for $\text{Hom}_{pro}(UP, UQ)$?

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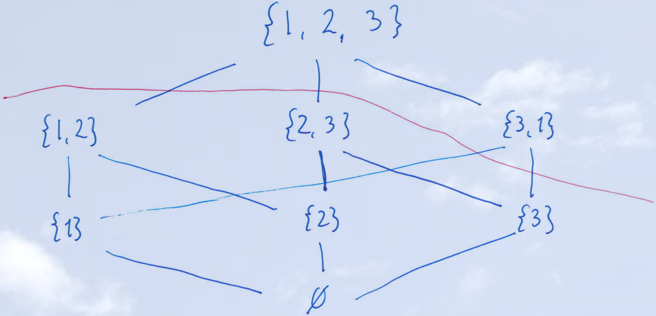
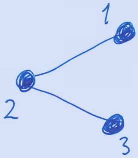
$$\mathrm{Hom}_{pro}(P,Q)\widehat{\ } \to \mathrm{Hom}_{pro}(UP,Q)\widehat{\ } \to \mathrm{Hom}_{pro}(UP,UQ)\widehat{\ }$$

Cuts in Boolean posets

$\overset{1-1}{\leftrightarrow}$ simplicial complexes



- Cut ~ gives simplicial complex:



Application I

Lots of spheres

Let $Q = [n]$ so $\widehat{Q} = [n+1]$.

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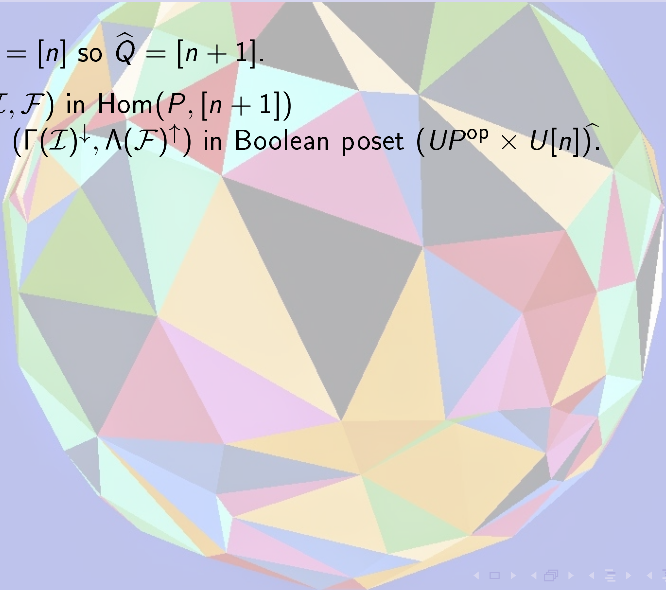

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This corresponding simplicial complex is a triangulated ball. Its boundary has a simple compact description and is a simplicial sphere.

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Two previously studied cases:

- When P is antichain and $n = 1$: Bier spheres (A.Björner, G.Ziegler et.al.)
- When P is a chain $[m]$: Squeezed spheres (G.Kalai)

Application II

Ideals in polynomial rings: Strongly stable ideals

$$\mathrm{Hom}_{pro}(\mathbb{N}, \mathbb{N}) = \mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}}) = \mathrm{Hom}(\mathbb{N}, \mathbb{N} \cup \infty).$$

- Self-dual poset (another advantage of profuntors!)
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One-to-one correspondence (must use Γ or Λ):

- Open poset ideals $\mathcal{I}$ in $\mathrm{Hom}_{pro}(\mathbb{N}, \mathbb{N})$
- Strongly stable ideals in infinite dimensional polynomial ring $k[x_1, x_2, \ldots, x_n, \cdots]$

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One-to-one correspondence:

- "Dedekind cuts" $(\mathcal{I}, \mathcal{F})$ in $\text{Hom}_{pro}(\mathbb{N}, \mathbb{N})$,
- *Dualizable* strongly stable ideals in $k[x_1, \ldots, x_n, \ldots]$.

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Thank you!