Restricting Power: Pebble-relation comonad in finite model theory

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Let σ be a set of relational symbols with positive arities, we can define a category of σ-structures $\mathcal{R}(\sigma)$:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R\in\sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- If there exists a morphism $f : \mathcal{A} \to \mathcal{B}$, we write $\mathcal{A} \to \mathcal{B}$

Category theorists look at structures "as they really are"; i.e. up to isomorphism $\mathcal{A} \cong \mathcal{B}$

Model theorists look at structures with "fuzzy glasses" imposed by a logic $\mathcal{L}$:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Leftrightarrow \mathcal{B} \vDash \phi$$

$$\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \equiv^{\mathcal{L}} \mathcal{B}$$

Used to study what properties are inexpressible in $\mathcal{L}$

To show P inexpressible in $\mathcal{L}$, define $\mathcal{A}, \mathcal{B}$ where $P(\mathcal{A})$ and not $P(\mathcal{B})$. Must show that $\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B}$

Over finite structures, $\equiv^{\mathbf{FOL}}$ is the same as $\cong$

Finite model theorists look at structures with a "fuzzy phoropter" imposed by grading a logic:

- Quantifier rank $\leq n$, QR_n
- Restrict number of variables be $\leq k$, $\mathcal{V}^k$

$$\phi = \exists x_1(\exists x_2(E(x_1, x_2) \wedge \exists x_3 E(x_3, x_2)) \wedge \forall x_4 E(x_1, x_4))$$

$\phi \in QR_3$ and $\phi \in \mathcal{V}^4$

To show P inexpressible in $\mathcal{L}$ over the **finite**, define $\mathcal{A}_k, \mathcal{B}_k$ for every k where $P(\mathcal{A}_k)$ and not $P(\mathcal{B}_k)$. Must show that $\mathcal{A}_k \equiv^{\mathcal{L}_k} \mathcal{B}_k$

CSP: Find assignment of variables $\mathcal{A}$ to a domain of values $\mathcal{B}$ satisfying a set of constraints, which can be encoded as relations on $\mathcal{B}$

A CSP can be formulated in $\mathcal{R}(\sigma)$ as deciding if there exists a morphism $h : \mathcal{A} \to \mathcal{B}$

Non-uniform problem $\mathsf{CSP}(\mathcal{B})$: fixing the set of values $\mathcal{B}$ and varying the variables $\mathcal{A}$.

In general, $\mathsf{CSP}(\mathcal{B})$ is NP-complete

Tractable cases of $\mathsf{CSP}(\mathcal{B})$ can be identified by considering approximations to homomorphism

Equivalence in a logic with parameter k approximates isomorphism:

$$\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B}$$

Preservation in the existential-positive fragment is an approximation to homomorphism:

$$\mathcal{A} \to \mathcal{B} \Rightarrow \mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B}$$

$$\mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B} \Leftrightarrow \forall \phi \in \exists^+\mathcal{L}_k, \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi$$

We will consider the existential-positive fragment of k-variable logic $\exists^+\mathcal{V}_k$

For all finite $\mathcal{A}$,

$$\mathcal{A} \Rrightarrow^{\exists^+\mathcal{V}^k} \mathcal{B} \to \mathcal{A} \to \mathcal{B}$$

then $\mathcal{B}$ has *k-treewidth duality*

$\mathcal{B}$ has k-treewidth duality $\Rightarrow \mathsf{CSP}(\mathcal{B}) \in \mathbf{PTIME}$

Proposition

The following are equivalent:

- $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{V}^k} \mathcal{B}$
- *Duplicator has a winning strategy in a forth k -pebble game*
- *For all finite $\mathcal{C}$ w/ treewidth $< k$, $\mathcal{C} \to \mathcal{A} \Rightarrow \mathcal{C} \to \mathcal{B}$*

- Spoiler and Duplicator each have k pebbles. On each round of $\exists^+\mathbf{Peb}_k(\mathcal{A},\mathcal{B})$:
  - Spoiler places his pebble $p \in \mathbf{k}$ on an element $a_i \in \mathcal{A}$
    - If p was already placed, Spoiler moves the pebble.
  - Duplicator places her corresponding pebble $p \in \mathbf{k}$ on $b_i \in \mathcal{B}$

  Duplicator wins if

$$\gamma = \{(a,b) \mid p \in \mathbf{k} \text{ w/ } p \text{ pebbling } a \in \mathcal{A}, b \in \mathcal{B}\ \}$$

  is a partial homomorphism

If Duplicator can always produce a winning move for any choice made Spoiler, than Duplicator has a winning strategy

Theorem ([KV90])

Duplicator has a winning strategy in $\exists^+\mathbf{Peb}_k(\mathcal{A}, \mathcal{B})$ *iff* $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{V}^k} \mathcal{B}$

Intuition:

$$\mathcal{A} \vDash \exists x_p \phi(x_p, \bar{y}) \Rightarrow \mathcal{A} \vDash \phi(a/x_p, \bar{y})$$

Spoiler places p on witness $a \in A$

Suppose Duplicator responds by putting p on $b \in B$

Partial homomorphism in winning condition $\Rightarrow$

$$\mathcal{B} \vDash \phi(b/x_p, \bar{y}) \Rightarrow \mathcal{B} \vDash \exists x_p \phi(x_p, \bar{y})$$

Forth k-pebble game

Intuitively, Spoiler is moving a k-sized window around the structure $\mathcal{A}$ during a play

Duplicator than has to choose a homomorphism from the k-sized window into $\mathcal{B}$

If Duplicator can't produce such a partial homomorphism than Spoiler wins

The k sized window is local 'view' of the structure

We can ‘internalize’ $\exists^{+}\mathbf{Peb}_k$ game by encoding it as a comonad $\mathbb{P}_k$, for every k, over $\mathcal{R}(\sigma)$

Suprisingly: we are also able to define the combinatorial parameter treewidth using coalgebrs of $\mathbb{P}_k$

Pebbling comonad [ADW17, AS18]

Given a σ-structure $\mathcal{A}$, we can create σ-structure on the set of Spoiler moves $\mathbb{P}_k A$ in $\exists^+\mathbf{Peb}_k(\mathcal{A}, \cdot)$, i.e. non-empty sequences of pairs (p, a) where $p \in \mathbf{k} = \{1, \dots, k\}$ and $a \in A$

Let $\varepsilon_{\mathcal{A}} : \mathbb{P}_k\mathcal{A} \to \mathcal{A}$ be $[(p_1, a_1), \dots, (p_n, a_n)] \mapsto a_n$ and $\pi_{\mathcal{A}} : \mathbb{P}_k\mathcal{A} \to \mathbf{k}$ be $[(p_1, a_1), \dots, (p_n, a_n)] \mapsto p_n$.

$$
\begin{aligned}
R^{\mathbb{P}_k\mathcal{A}}(s_1, \dots, s_r) \Leftrightarrow s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \text{ for } i, j \in \mathbf{r} \\
\text{and } \pi_{\mathcal{A}}(s_i) \text{ does not appear in } \mathsf{suffix}(s_i, s) \\
\text{where } s = \max(s_1, \dots, s_r) \\
\text{and } R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s_1), \dots, \varepsilon_{\mathcal{A}}(s_r))
\end{aligned}
$$

For $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$ define $f^* : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathcal{B}$ recursively:

$$f^*(s[(p, a)]) = f^*(s)[f(s[(p, a)])]$$

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- Functions $f : \mathbb{P}_k A \to B$ are Duplicator's strategies in $\exists^+\mathbf{Peb}(\mathcal{A}, \mathcal{B})$
- Chose relations so that σ-morphisms $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$ are Duplicator's **winning** strategies.
- Coextension $f^* : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathcal{B}$ models history preservation of the game

Theorem ([ADW17])

The following are equivalent:

1. *Duplicator has a winning strategy in* $\exists^+\mathbf{Peb}(\mathcal{A}, \mathcal{B})$
2. *There exists a coKleisli morphism* $f : \mathbb{P}_k\mathcal{A} \to \mathcal{B}$

Can be strengthened to a bijective correspondence using relative comonads and explicit equality in signature

Another characterization of this ‘k-approximate homomorphism relation’

Proposition

The following are equivalent:

- $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{V}^k} \mathcal{B}$
- *Duplicator has a winning strategy in* $\exists^+\mathbf{Peb}_k(\mathcal{A}, \mathcal{B})$
- *For all finite* $\mathcal{C}$ *w/ treewidth* $< k$, $\mathcal{C} \to \mathcal{A} \Rightarrow \mathcal{C} \to \mathcal{B}$
- *There exists a Kleisli morphism* $\mathbb{P}_k\mathcal{A} \to \mathcal{B}$

Eilenberg-Moore category of coalgebras

We want to use coalgebras of $\mathbb{P}_k$ to define treewidth

Coalgebras are morphisms $\alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A}$ satisfying the equations:

$$\epsilon_{\mathcal{A}} \circ \alpha = \mathsf{id}_{\mathcal{A}} \qquad \mathbb{C}_k\alpha \circ \alpha = \delta_{\mathcal{A}} \circ \alpha$$

with $\delta_{\mathcal{A}} = \mathsf{id}^*_{\mathbb{P}_k\mathcal{A}} : \mathbb{P}_k\mathcal{A} \to \mathbb{P}_k\mathbb{P}_k\mathcal{A}$

We can define the Eilenberg-Moore category $\mathcal{EM}(\mathbb{P}_k)$:

- Objects are coalgebras $(\mathcal{A}, \alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A})$
- Morphisms are commuting squares:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathbb{P}_k\mathcal{A} \arrow[d, "\mathbb{P}_k f"] \\ \mathcal{B} \arrow[r, "\beta"] & \mathbb{P}_k\mathcal{B} \end{tikzcd}$$

For every structure $\mathcal{A}$, define the Gaifman graph $\mathcal{G}(\mathcal{A})$ w/ vertices A and

$a \frown a' \in \mathcal{G}(\mathcal{A}) \Leftrightarrow a = a'$ or a, a' appear in some tuple of $R^{\mathcal{A}}$

Intuition: Treewidth $\mathsf{tw}(\mathcal{A})$ measures how far $\mathcal{G}(\mathcal{A})$ is from being a tree

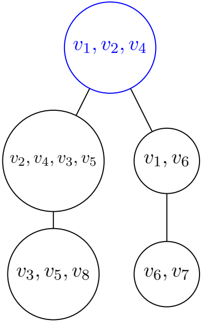
Often implicit in dynamic programming algorithms, i.e k-consistency algorithms

Formally: Treewidth is the minimum width of a tree-decomposition of $\mathcal{G}(\mathcal{A})$

Definition

A tree decomposition of $\mathcal{A}$ of width k is a triple $(T, \leq_T, \lambda : T \to \mathcal{P}A)$

- Every $a \in \mathcal{A}$ is in some node of T
- All the nodes containing $a \in \mathcal{A}$ form a subtree
- For every $a \frown a' \in \mathcal{G}(\mathcal{A})$, $\{a, a'\} \subseteq \lambda(x)$
- $k = \max\{|\lambda(x)|\}_{x \in T} - 1$



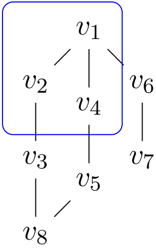
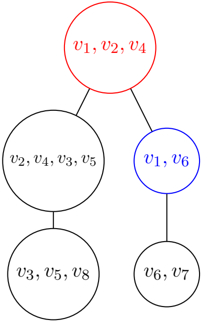
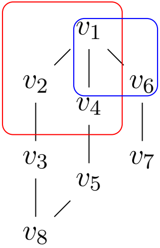


Figure: Tree decomposition of width 3 for $\mathcal{G}(\mathcal{A})$





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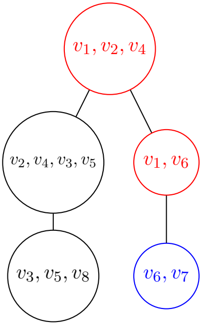

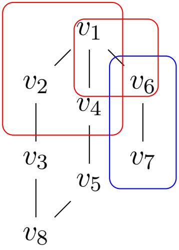

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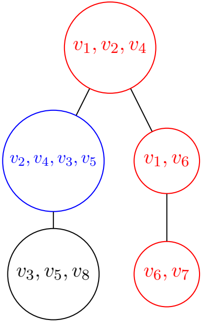

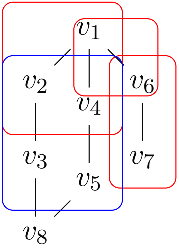

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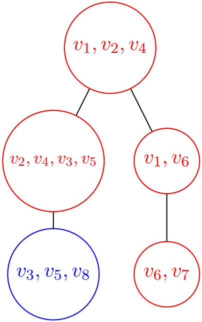

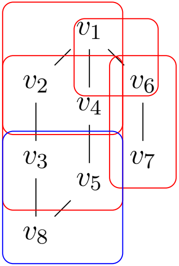

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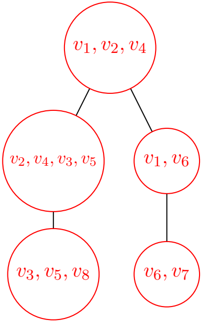

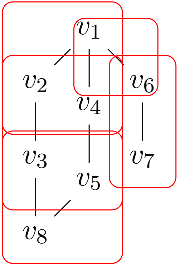

Figure: Tree decomposition of width 3 for $\mathcal{G}(\mathcal{A})$

We can define a category of k-pebble forest covers $\mathcal{F}(\sigma)^k$, where objects $(\mathcal{A}, \leq, p : \mathcal{A} \to \mathbf{k})$ satisfying:

- All elements below $a \in \mathcal{A}$ in $\leq$ form a chain
- If $a \frown a' \in \mathcal{G}(\mathcal{A})$, $a \leq a'$ or $a' \leq a$
- If $a \frown a'$ and $a \leq a'$, then for all b with $a < b \leq a'$, $p(a) \neq p(b)$

Morphisms are functions that preserve immediate successors in the order $\leq$ and the pebbling function

$\mathbb{P}_k$ arises from the comonadic adjunction $U^k \dashv F^k$ where
$U^k : \mathcal{F}(\sigma)^k \to \mathcal{R}(\sigma)$, $F^k\mathcal{A} = (\mathbb{P}_k\mathcal{A}, \sqsubseteq, \pi_{\mathcal{A}})$

Theorem ([AM20])

The category of coalgebras $\mathcal{EM}(\mathbb{P}_k)$ *is isomorphic to* $\mathcal{F}(\sigma)^k$

Theorem ([ADW17, AS18])

The following are equivalent:

1. $\mathcal{A}$ *has a tree decomposition of width* $< k$
2. $\mathcal{A}$ *has a* k*-pebble forest cover, i.e. coalgebra* $\mathcal{A} \to \mathbb{P}_k\mathcal{A}$

Let $\kappa^{\mathbb{C}}(\mathcal{A})$ be the least k such that there exists coalgebra $\mathcal{A} \to \mathbb{C}_k\mathcal{A}$

Corollary ([ADW17])

$\kappa^{\mathbb{P}}(\mathcal{A}) = \mathsf{tw}(\mathcal{A}) + 1$

What about pathwidth?

We say a tree decomposition $(T, \leq, \lambda)$ of $\mathcal{A}$ is a *path decomposition* if $\leq$ is a linear order

Pathwidth $\mathsf{pw}(\mathcal{A})$ is the minimum width of a path decomposition of $\mathcal{A}$

Closely linked to CSPs in **NLOGSPACE** analogous to treewidth's relationship to **PTIME**

Is there an analogous comonad to $\mathbb{P}_k$, but for pathwidth?

Pebble-relation comonad

Given a σ-structure $\mathcal{A}$, we can create σ-structure $\mathbb{PR}_k\mathcal{A}$ on the set of pairs $([(p_1, a_1), \dots, (p_n, a_n)], i)$ with $i \in \mathbf{n}$

- $\varepsilon_{\mathcal{A}} : \mathbb{PR}_k\mathcal{A} \to \mathcal{A}$ be $([(p_1, a_1), \dots, (p_n, a_n)], i) \mapsto a_i$
- $\pi_{\mathcal{A}} : \mathbb{PR}_k\mathcal{A} \to \mathbf{k}$ be $([(p_1, a_1), \dots, (p_n, a_n)], i) \mapsto p_i$.
- For $i < j$, $s(i, j]$ is the subsequence of s starting at $i + 1$ and ending at j (inclusive)

$$\begin{aligned} R^{\mathbb{P}_k\mathcal{A}}((s, i_1), \dots, (s, i_r)) \Leftrightarrow\ & \pi_{\mathcal{A}}(s, i_j) \text{ does not appear in } s(i_j, m] \\ & \text{where } m = \max(i_1, \dots, i_j) \\ & \text{and } R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s, i_1), \dots, \varepsilon_{\mathcal{A}}(s, i_r)) \end{aligned}$$

Let $s = [(p_1, a_1)], \dots, (p_n, a_n)] \in \mathbb{PR}_k\mathcal{A}$ and $f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B}$

$$f^*(s, i) = [(p_1, f(s, 1)), \dots, (p_n, f(s, n))], i)$$

Pathwidth and $\mathbb{PR}_k$

We can define a subcategory $\mathcal{LF}(\sigma)^k$ of the k-pebble forest covers $\mathcal{F}(\sigma)^k$ where the forests are linear forests

$\mathbb{PR}_k$ arises from the comonadic adjunction $U^k \dashv L^k$ where $U^k : \mathcal{LF}(\sigma)^k \to \mathcal{R}(\sigma)$, $L^k\mathcal{A} = (\mathbb{PR}_k\mathcal{A}, \leq^*, \pi_{\mathcal{A}})$

$$(t, i) \leq^* (t', j) \Leftrightarrow t = t' \text{ and } i \leq j$$

Theorem ([AM20])

The category of coalgebras $\mathcal{EM}(\mathbb{PR}_k)$ is isomorphic to $\mathcal{LF}(\sigma)^k$

Theorem

The following are equivalent:

1. $\mathcal{A}$ *has a path decomposition of width* $< k$
2. $\mathcal{A}$ *has a* k*-pebble linear forest cover, i.e. coalgebra* $\mathcal{A} \to \mathbb{PR}_k\mathcal{A}$

Corollary

$\kappa^{\mathbb{PR}}(\mathcal{A}) = \mathsf{pw}(\mathcal{A}) + 1$

Definition ([Dal05])

Restricted conjunction fragment $\exists^+\mathcal{N}_k \subseteq \exists^+\mathcal{V}_k$ where conjunctions $\bigwedge \Psi$ have that Ψ:

- At most one formula in Ψ containing quantifiers has a free variable.

Theorem ([Dal05])

The following are equivalent:

- $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{N}^k} \mathcal{B}$
- *Duplicator has a winning strategy in a k pebble relation game* $\exists^+\mathbf{PebR}_k(\mathcal{A},\mathcal{B})$
- *For all* $\mathcal{C}$ *w/ pathwidth* $< k$, $\mathcal{C} \to \mathcal{A} \Rightarrow \mathcal{C} \to \mathcal{B}$

Forth k pebble-relation game

The k pebble-relation game is cumbersome to state formally

- Spoiler chooses a at most k sized window on the structure $\mathcal{A}$ (as in the k-pebble game)
- Duplicator responds with a **set** of homomorphisms from that window into $\mathcal{B}$ (non-determinism)
- Response set must extend some of the partial homomorphisms of her previous move
- Spoiler wins if Duplicator can only respond with the empty set

We can interpret elements of $\mathbb{PR}_k\mathcal{A}$ as Spoiler plays, in some new game

This produces a simpler equivalent game: preannounced or all-in-one k-pebble game

Pre-announced existential k-pebble game

The pre-announced k-pebble game $\exists^+ \mathbf{PPeb}_k(\mathcal{A}, \mathcal{B})$ is played in one round:

- Spoiler chooses a list of k-pebble placements on $\mathcal{A}$:

$$s = [(p_1, a_1), \ldots, (p_n, a_n)]$$

- Duplicator chooses a compatible list of k-pebble placements on $\mathcal{B}$:

$$t = [(p_1, b_1), \ldots, (p_n, b_n)]$$

Duplicator wins if for every index $i \in \mathbf{n}$, the pairs of pebble placements in $s(0, i]$ and $t(0, i]$ form a partial homomorphism.

Stewart's all-in-one existential k-pebble game [Ste07]

Proposition

The following are equivalent:

- $\mathcal{A} \Rrightarrow^{\exists^+\mathcal{N}^k} \mathcal{B}$
- *Duplicator has a winning strategy in* $\exists^+\mathbf{PebR}_k(\mathcal{A}, \mathcal{B})$
- *For all finite* $\mathcal{C}$ *w/ pathwidth* $< k$, $\mathcal{C} \to \mathcal{A} \Rightarrow \mathcal{C} \to \mathcal{B}$
- *There exists* $f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B}$
- *Duplicator has a winning strategy in* $\exists^+\mathbf{PPeb}_k(\mathcal{A}, \mathcal{B})$

Bounded parameter duality

Definition

A structure $\mathcal{B}$ has the $\mathbb{C}_k$-*lifting property* if for every structure $\mathcal{A}$:

$$\mathbb{C}_k\mathcal{A} \to \mathcal{B} \Rightarrow \mathcal{A} \to \mathcal{B}$$

$\mathcal{B}$ has k-treewidth duality iff $\mathcal{B}$ has the $\mathbb{P}_k$-lifting property.

$\mathcal{B}$ has k-pathwidth duality iff $\mathcal{B}$ has the $\mathbb{PR}_k$-lifting property.

$\mathcal{B}$ has k-treewidth duality for some $k \Rightarrow \mathsf{CSP}(\mathcal{B}) \in \mathbf{P}$[DKV02] (converse does not hold [Ats08])

$\mathcal{B}$ has k-pathwidth duality for some $k \Rightarrow \mathsf{CSP}(\mathcal{B}) \in \mathbf{NL}$[Dal05] (converse open, but hard)

Summary table

$\mathbb{C}_k$	**Logic**	$\kappa^{\mathbb{C}}$	$\rightarrow_k^{\mathbb{C}}$	$\leftrightarrow_k^{\mathbb{C}}$	$\cong_k^{\mathbb{C}}$
$\mathbb{E}_k$ [AS18]	**FOL** w/ $\mathsf{qr} \leq k$	tree-depth	✓	✓	✓
$\mathbb{P}_k$ [ADW17]	k-variable logic	treewidth $+1$	✓	✓	✓
$\mathbb{M}_k$ [AS18]	**ML** w/ $\mathsf{md} \leq k$	sync. tree-depth	✓	✓	✓
$\mathbb{G}_k^{\mathfrak{g}}$ [AM20]	$\mathfrak{g}$-guarded logic w/ width $\leq k$	guarded treewidth	✓	✓	?
$\mathbb{H}_{n,k}$ [CD20]	k-variable logic w/ $\mathbf{Q}_n$-quantifiers	n-ary general treewidth	✓	✓	✓
$\mathbb{PR}_k$	k-variable logic restricted-$\wedge$	pathwidth $+1$	✓	?	✓
$\mathbb{LG}_k$	k-conjunct guarded logic	hypertree-width	✓	?	?

Theorem

1. $\mathcal{A} \rightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \mathcal{A} \Rrightarrow^{\exists^+\mathcal{L}_k} \mathcal{B} \Leftrightarrow$ *Duplicator wins* $\exists^+\mathbf{G}_k(\mathcal{A}, \mathcal{B})$
2. $\mathcal{A} \leftrightarrow_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv^{\mathcal{L}_k} \mathcal{B} \Leftrightarrow$ *Duplicator wins* $\mathbf{G}_k(\mathcal{A}, \mathcal{B})$
3. $\mathcal{A} \cong_k^{\mathbb{C}} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} \Leftrightarrow$ *Duplicator wins* $\#\mathbf{G}_k(\mathcal{A}, \mathcal{B})$

The $\rightarrow_k^{\mathbb{C}}$ and $\cong_k^{\mathbb{C}}$ arise from $\mathcal{K}(\mathbb{C}_k)$

The $\leftrightarrow_k^{\mathbb{C}}$ arises from a notion of open map bisimulation in the category of coalgebras over $\mathbb{C}_k$

All structures finite

Theorem ([Lov67])

$\mathcal{A} \cong \mathcal{B} \Leftrightarrow \mathsf{Hom}(\mathcal{C}, \mathcal{A}) \cong \mathsf{Hom}(\mathcal{C}, \mathcal{B})$ *for* $\mathcal{C}$

Theorem ([Gro20])

$\mathcal{A} \equiv^{QR_n(\#)} \mathcal{B} \Leftrightarrow \mathsf{Hom}(\mathcal{C}, \mathcal{A}) \cong \mathsf{Hom}(\mathcal{C}, \mathcal{B})$ *for* $\mathcal{C}$ *w/* $\mathsf{td}(C) \leq n$

Theorem ([Dvo09])

$\mathcal{A} \equiv^{\mathcal{V}^k(\#)} \mathcal{B} \Leftrightarrow \mathsf{Hom}(\mathcal{C}, \mathcal{A}) \cong \mathsf{Hom}(\mathcal{C}, \mathcal{B})$ *for* $\mathcal{C}$ *w/* $\mathsf{tw}(\mathcal{C}) < k$,

Theorem ([DJR21])

$\mathcal{A} \equiv^{\mathcal{L}_k(\#)} \mathcal{B} \Leftrightarrow \mathsf{Hom}(\mathcal{C}, \mathcal{A}) \cong \mathsf{Hom}(\mathcal{C}, \mathcal{B})$ *for* $\mathbb{C}_k$*-coalgebras* $\mathcal{C}$

Spoiler-Duplicator game comonads unify and generalize the use of resource measures in finite model theory

These comonads are robustly defined, i.e. via a model-comparison game or a forest cover/decomposition

$\mathbb{PR}_k$ extends this framework to link pathwidth and a restricted conjunction fragment of k-variable logic $\exists^+\mathcal{N}_k$

Provides interesting avenues towards applying category theory to complexity theory:

$\mathcal{B}$ has the $\mathbb{PR}_k$-lifting property for some $k \Rightarrow \mathsf{CSP}(\mathcal{B}) \in \mathbf{NL}$

Samson Abramsky, Anuj Dawar, and Pengming Wang.
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