A Canonical Algebra of Open Transition Systems

Elena Di Lavore, Alessandro Gianola, Mario Román, Nicoletta Sabadini
and Paweł Sobociński

Applied Category Theory 2021
University of Cambridge.

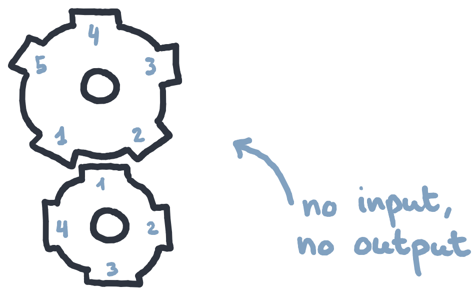
arXiv: 2010.10069

PART 1:

SPANS OF GRAPHS

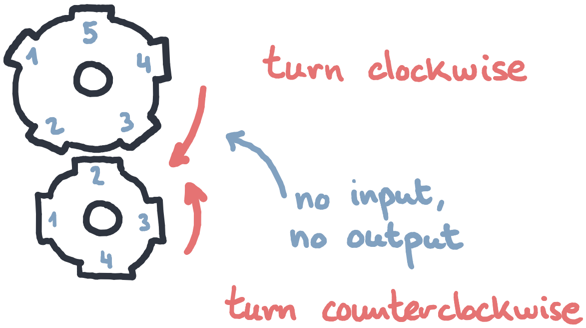
Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



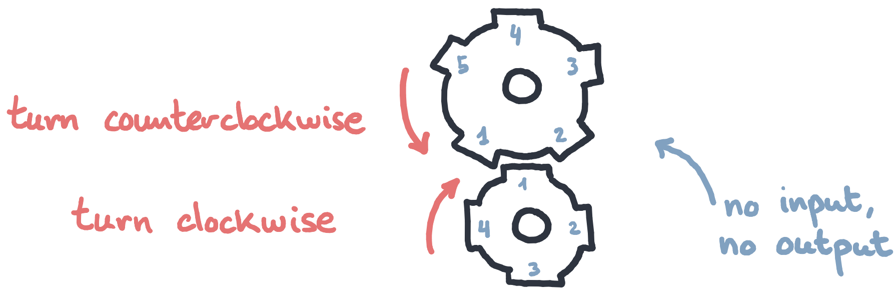
Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



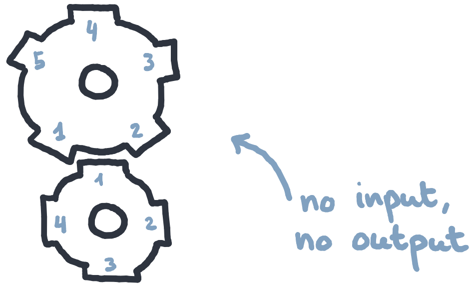
Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



Span(Graph), algebra of open transition systems

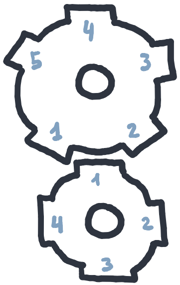
Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



How to model this situation?

Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.

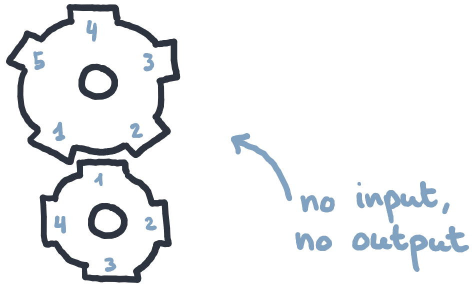


no input, no output

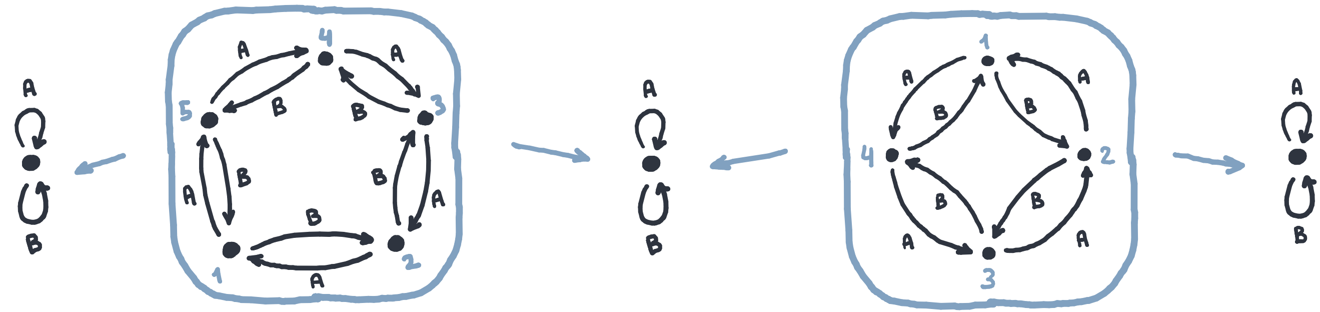
How to model this situation?

Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.

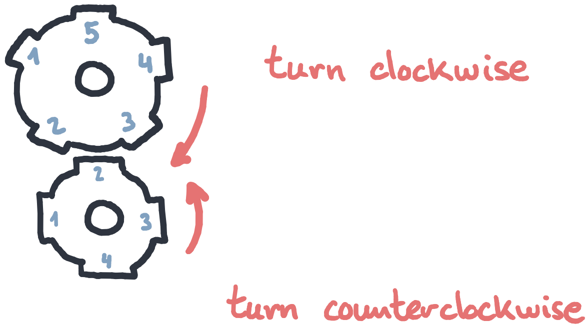


How to model this situation?

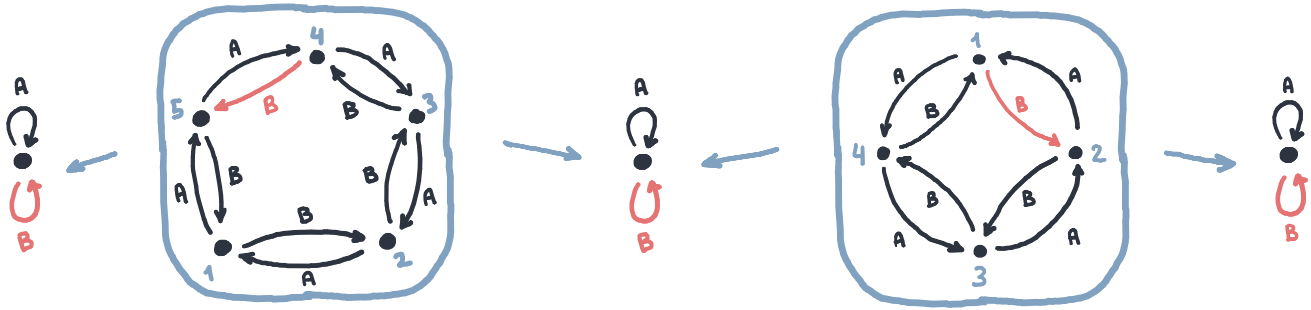


Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.

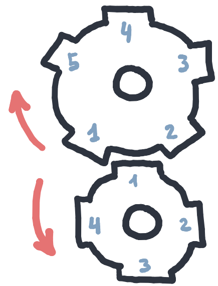


How to model this situation?

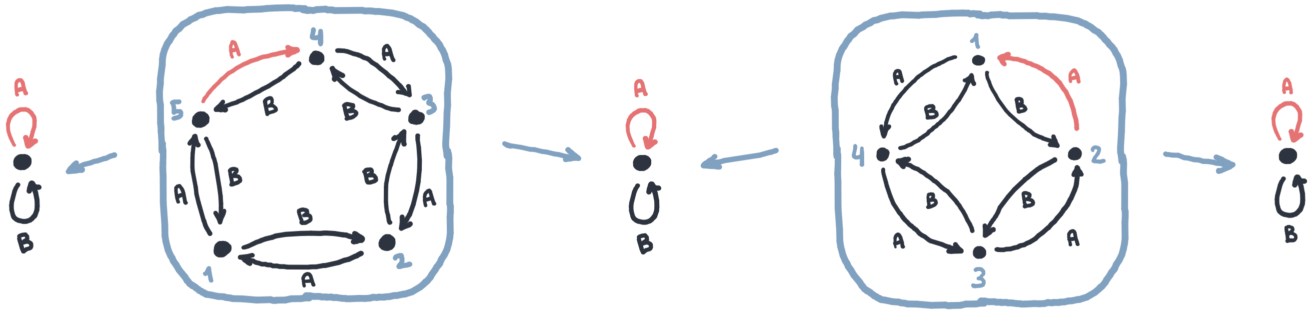


Span(Graph), algebra of open transition systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.

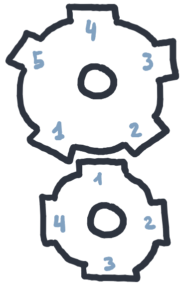


How to model this situation?

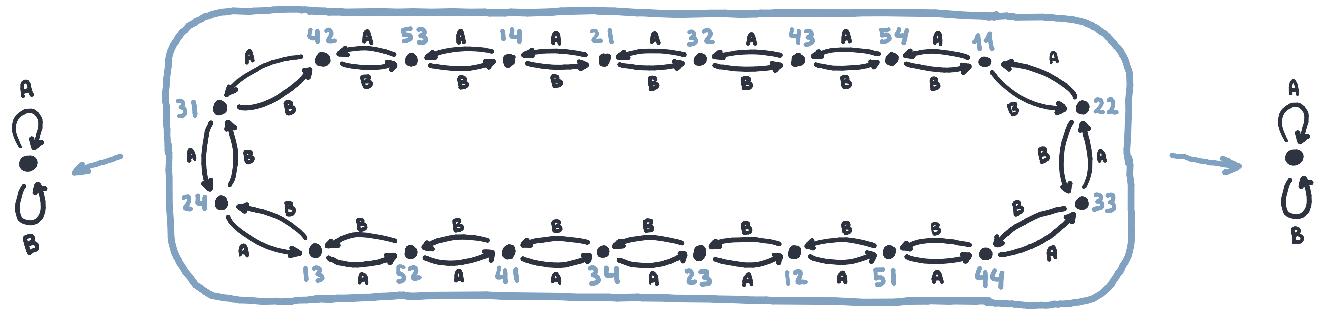


Span(Graph), algebra of open transition systems

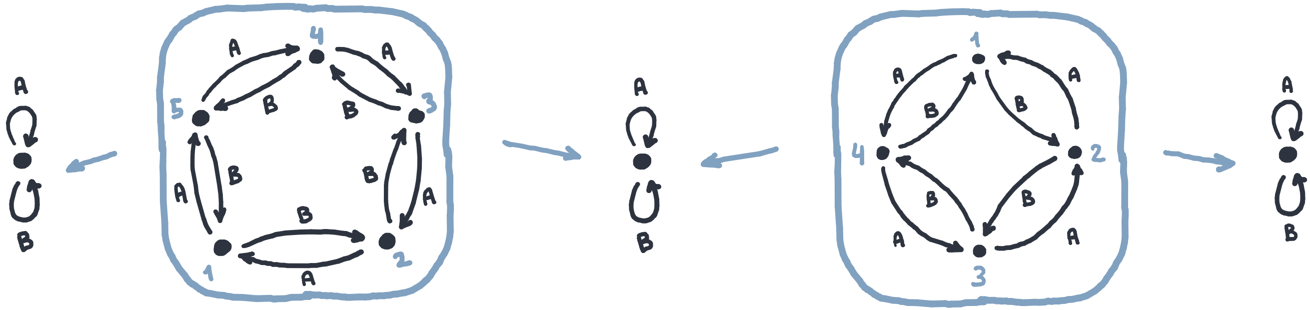
Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



How to model this situation?



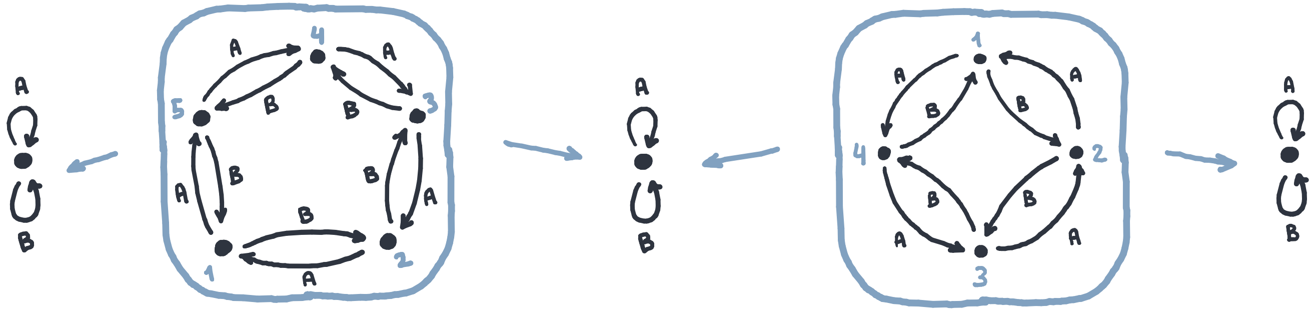
Span(Graph), algebra of open transition systems



SPAN (GRAPH):

- Compositional, stateful transition systems.
- Synchronization by composition.
- Transition systems are encoded as graphs.
- Boundaries may be single-vertex graphs, SPAN(GRAPH)$_*$.

Span(Graph), algebra of open transition systems



SPAN(GRAPH):

- Compositional, stateful transition systems.
- Synchronization by composition.
- Transition systems are encoded as graphs.

Ad hoc?

- Boundaries may be single-vertex graphs, $\text{SPAN}(\text{GRAPH})_*$.

PART 2:

STATEFUL MORPHISMS

The St(·) construction

DEFINITION. For $(\mathbb{C}, \otimes, I)$ symmetric monoidal,

$$\mathrm{St}(\mathbb{C})(A,B) := \Big\{ (\underset{\text{state space}}{S}, \overset{\text{stateful morphism}}{\varphi}) \;\Big|\; S \in \mathrm{ob}\,\mathbb{C},\ \varphi : S \otimes A \longrightarrow S \otimes B \Big\} /\sim,$$

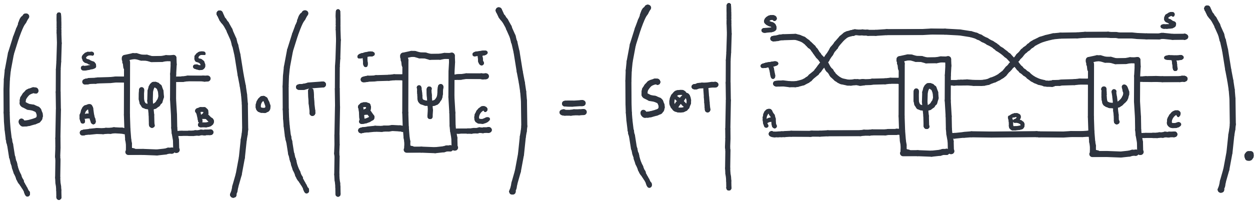
quotiented by the equivalence relation

where $\gamma : S \cong T$ is any isomorphism.

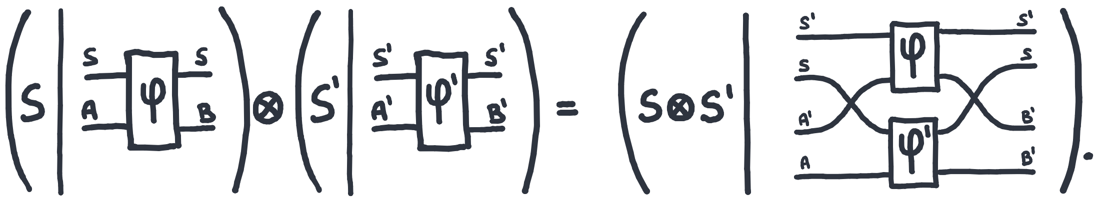
Diagrammatic algebra: from linear to concurrent systems. Bonchi, Holland, et al.
Memoryful geometry of interaction. Hoshino, Muroya, Hasuo.
Differentiable Causal Computations via Delayed Trace. Katsumata, Sprunger.

The St(·) construction

Composition is given by:



Tensoring is given by:



Universal property?

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

The $\mathrm{St}(\cdot)$ construction

$\mathrm{St}(\mathrm{Set}_\times)$: $S \times A \to S \times B$ Mealy transition system

$\mathrm{St}(\mathrm{Set}_+)$: $S + A \to S + B$ Elgot transition system

$\mathrm{St}(\mathrm{Rel}_\times)$: $S \times A \to P(S \times B)$ Non-deterministic transition system

The $\mathrm{St}(\cdot)$ construction

$\mathrm{St}(\mathrm{Set}_\times)$: $S \times A \to S \times B$ Mealy transition system

$\mathrm{St}(\mathrm{Set}_+)$: $S + A \to S + B$ Elgot transition system

$\mathrm{St}(\mathrm{Rel}_\times)$: $S \times A \to \mathcal{P}(S \times B)$ Non-deterministic transition system

$\mathrm{St}(\mathrm{Span}(\mathrm{Set}))$: $S \times A \leftarrow E \to S \times B$

The $\mathrm{St}(\cdot)$ construction

$\mathrm{St}(\mathrm{Set}_\times)$: $S \times A \to S \times B$ Mealy transition system

$\mathrm{St}(\mathrm{Set}_+)$: $S + A \to S + B$ Elgot transition system

$\mathrm{St}(\mathrm{Rel}_\times)$: $S \times A \to \mathcal{P}(S \times B)$ Non-deterministic transition system

$\mathrm{St}(\mathrm{Span}(\mathrm{Set}))$:

$$\begin{tikzcd} A & E \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & B \\ & S & \end{tikzcd}$$

The $\mathrm{St}(\cdot)$ construction

$\mathrm{St}(\mathrm{Set}_\times)$: $S \times A \to S \times B$ Mealy transition system

$\mathrm{St}(\mathrm{Set}_+)$: $S + A \to S + B$ Elgot transition system

$\mathrm{St}(\mathrm{Rel}_\times)$: $S \times A \to P(S \times B)$ Non-deterministic transition system

$\mathrm{St}(\mathrm{Span}(\mathrm{Set}))$:

$$\begin{tikzcd}
A \arrow[d, shift left] \arrow[d, shift right] & E \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[d, shift left] \arrow[d, shift right] \\
1 & S \arrow[l] \arrow[r] & 1
\end{tikzcd}$$

The St(·) construction

$\mathrm{St}(\mathrm{Set}_\times)$: $S \times A \longrightarrow S \times B$ Mealy transition system

$\mathrm{St}(\mathrm{Set}_+)$: $S + A \longrightarrow S + B$ Elgot transition system

$\mathrm{St}(\mathrm{Rel}_\times)$: $S \times A \longrightarrow P(S \times B)$ Non-deterministic transition system

$\mathrm{St}(\mathrm{Span}(\mathrm{Set}))$:

$$
\begin{tikzcd}
A \arrow[d, shift right] \arrow[d, shift left] & E \arrow[l] \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & B \arrow[d, shift right] \arrow[d, shift left] \\
1 & S \arrow[l] \arrow[r] & 1
\end{tikzcd}
$$

$\mathrm{Span}(\mathrm{Graph})_*$

The St(·) construction

$\mathsf{St}(\mathsf{Set}_{\times})$: $S \times A \to S \times B$ — Mealy transition system

$\mathsf{St}(\mathsf{Set}_{+})$: $S + A \to S + B$ — Elgot transition system

$\mathsf{St}(\mathsf{Rel}_{\times})$: $S \times A \to \mathcal{P}(S \times B)$ — Non-deterministic transition system

$\mathsf{St}(\mathsf{Span}(\mathsf{Set}))$:

$$\begin{tikzcd} A \arrow[d, shift left] \arrow[d, shift right] & E \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[d, shift left] \arrow[d, shift right] \\ 1 & S \arrow[l] \arrow[r] & 1 \end{tikzcd}$$

$\mathsf{Span}(\mathsf{Graph})_{*}$

Theorem. There is a monoidal isomorphism:

$$\mathsf{St}(\mathsf{Span}(\mathsf{Set})) \cong \mathsf{Span}(\mathsf{Graph})_{*}$$

stateful synchronization: spans of graphs

Part 3:

Feedback

Categories with feedback

Symmetric monoidal category with an operator

$$\mathsf{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A, B),$$

such that:

1. $u ; \mathsf{fbk}_S(f) ; v = \mathsf{fbk}_S((u \otimes id) ; f ; (v \otimes id))$
2. $\mathsf{fbk}_I(f) = f$
3. $\mathsf{fbk}_S(\mathsf{fbk}_T(f)) = \mathsf{fbk}_{S \otimes T}(f)$
4. $\mathsf{fbk}_S(f) \otimes g = \mathsf{fbk}_S(f \otimes g)$
5. $\mathsf{fbk}(f ; (h \otimes id)) = \mathsf{fbk}((h \otimes id) ; f)$

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

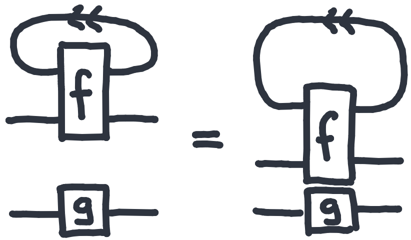
Symmetric monoidal category with an operator

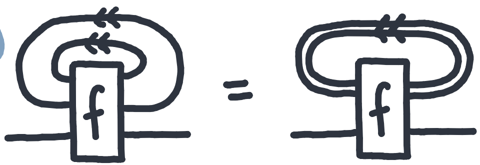
$$\mathrm{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A, B),$$

such that:

1


2


4


3


5


Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

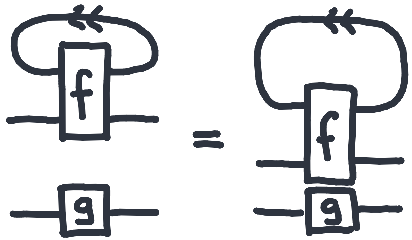
Categories with feedback

Symmetric monoidal category with an operator

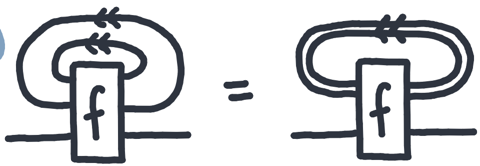
$$\mathrm{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A, B),$$

such that:

1


4


2


3


5

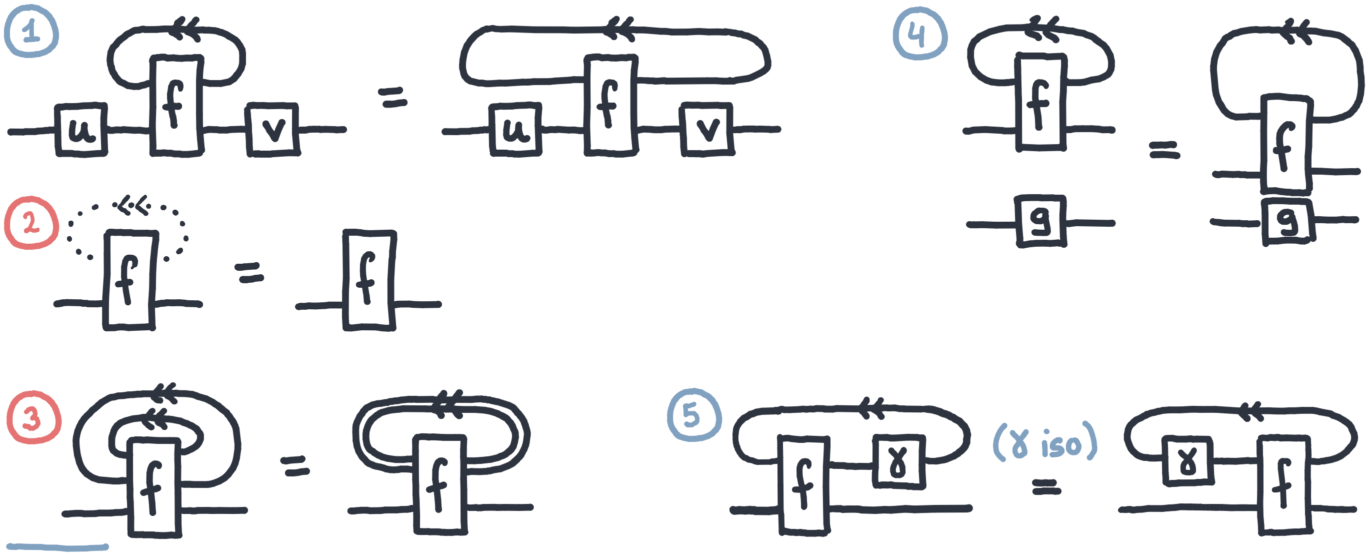

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

Symmetric monoidal category with an operator

$$\mathrm{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A,B),$$

such that:



Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

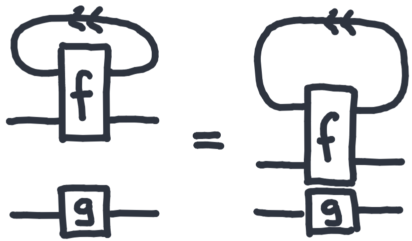
Symmetric monoidal category with an operator

$$\mathrm{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A, B),$$

such that:

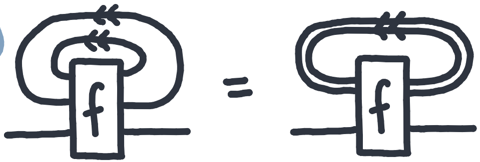
1

4



2

3



5

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

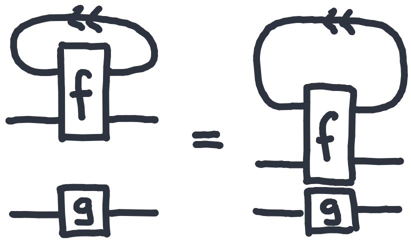
Categories with feedback

Symmetric monoidal category with an operator

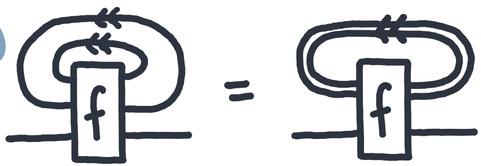
$$\mathrm{fbk}_S : \hom(S \otimes A, S \otimes B) \longrightarrow \hom(A, B),$$

such that:

1


4


2


3


5


Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

Differences with traced monoidal categories?

i.

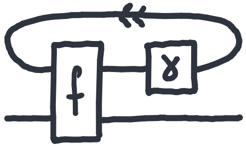
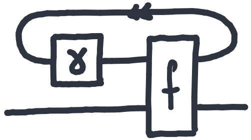
(WEAKSLIDING)

ii.

(YANKING)

Categories with feedback

Differences with traced monoidal categories?

i.  $\overset{(\gamma\ \text{iso})}{=}$  (WEAKSLIDING)

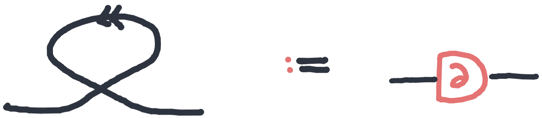
ii.  $\neq$  (YANKING)

Categories with feedback

Differences with traced monoidal categories?

i. 

(WeakSliding)

ii. 

(Yanking)

Categories with feedback

Differences with traced monoidal categories?

i. 

(WeakSliding)

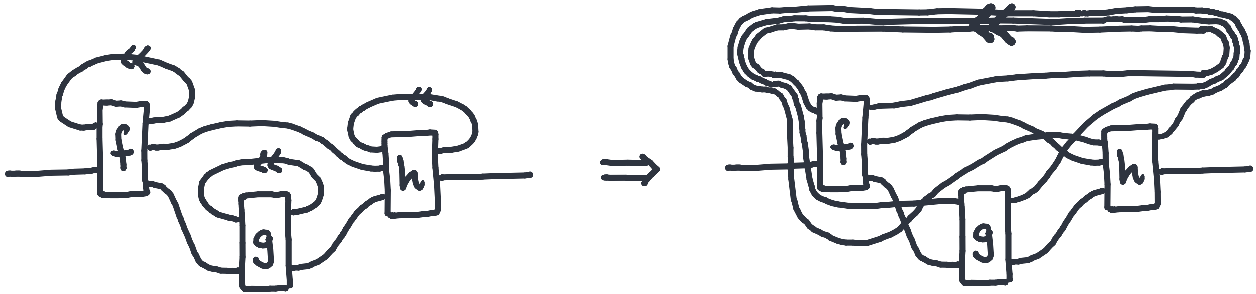
ii. 

(Yanking)

- Feedback is weaker than trace (and balanced trace).
- Feedback and guarded trace coincide in compact closed categories.
- Feedback has a different type than delayed trace.

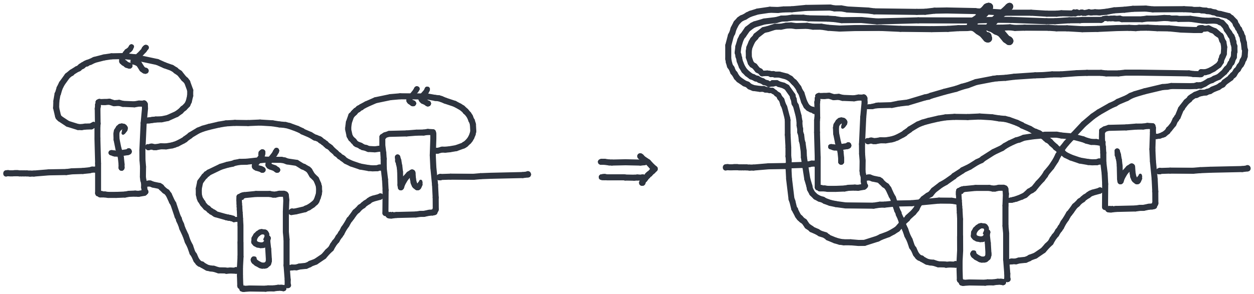
Categories with feedback

Multiple applications of feedback can be reduced into a single one. All of the axioms are needed for this result.



Categories with feedback

Multiple applications of feedback can be reduced into a single one. All of the axioms are needed for this result.

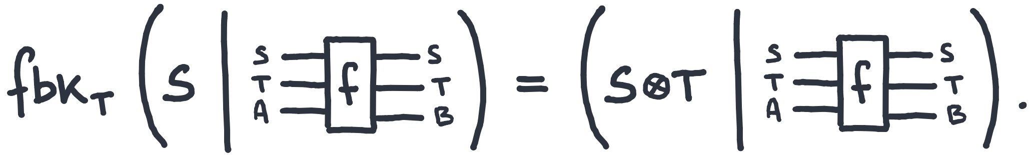


We can use this to show that $\mathrm{St}(\mathbb{C})$ is the free category with feedback.

$$\begin{tikzcd} \mathrm{SymMon} \arrow[rr, bend left, "\mathrm{St}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Feedback} \arrow[ll, bend left, "\mathcal{U}"] \end{tikzcd}$$

Categories with feedback

PROPOSITION. Let $\mathbb{C}$ be a symmetric monoidal category. $\mathsf{St}(\mathbb{C})$ has a feedback structure given by



THEOREM. Let $\mathbb{C}$ be a symmetric monoidal category. The symmetric monoidal category $\mathsf{St}(\mathbb{C})$ is the free category with feedback over $\mathbb{C}$, meaning that

$$
\begin{tikzcd}
\mathsf{St}(\mathbb{C}) \arrow[r, "\exists! \tilde{F}"] & \mathbb{D} \\
\mathbb{C} \arrow[u, "i"] \arrow[ur, bend right, "F"'] &
\end{tikzcd}
$$

Feedback-preserving: $\tilde{F}(\mathrm{fbk}_S(f)) = \mathrm{fbk}_{FS}(\tilde{F}(f))$.

Category with feedback.

Symmetric monoidal functor.

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

THEOREM. The following is an isomorphism of categories.

$$\mathsf{Span}(\mathsf{Graph})_* \cong \mathsf{St}(\mathsf{Span}(\mathsf{Set}))$$

Stateful

Synchronization

$\mathsf{Span}(\mathsf{Graph})_*$ is the free category with feedback over $\mathsf{Span}(\mathsf{Set})$.

Categories with feedback

THEOREM. The following is an isomorphism of categories.

$$\mathrm{Span}(\mathrm{Graph})_* \cong \mathrm{St}(\mathrm{Span}(\mathrm{Set}))$$

Stateful

Synchronization

$\mathrm{Span}(\mathrm{Graph})_*$ is the free category with feedback over $\mathrm{Span}(\mathrm{Set})$.

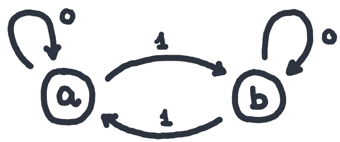
Example:

Set $\longrightarrow$ Span lifts to MealyTS $\longrightarrow$ $\mathrm{Span}(\mathrm{Graph})_*$.

$f : A \to B$

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

$f(a,0) = a$
$f(a,1) = b$
$f(b,0) = b$
$f(b,1) = a$



Categories with feedback

THEOREM. The following is an isomorphism of categories.

$$\mathrm{Span}(\mathrm{Graph})_* \cong \mathrm{St}(\mathrm{Span}(\mathrm{Set}))$$

Stateful   Synchronization

$\mathrm{Span}(\mathrm{Graph})_*$ is the free category with feedback over $\mathrm{Span}(\mathrm{Set})$.

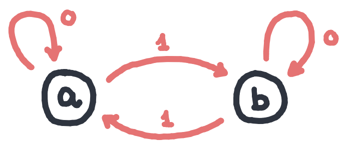
Example:

Set $\longrightarrow$ Span lifts to MealyTS $\longrightarrow$ $\mathrm{Span}(\mathrm{Graph})_*$.

$f : A \to B$

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

$f(a,0) = a$
$f(a,1) = b$
$f(b,0) = b$
$f(b,1) = a$



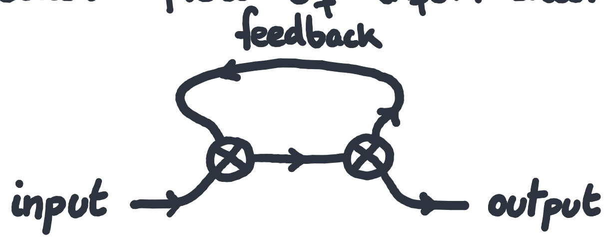
PART 4:

GENERALIZING $St(\cdot)$

Generalizing Feedback

Feedback describes a particular flow of information.

normal flow



flow with feedback

Generalizing Feedback

Feedback describes a particular flow of information.

normal flow | flow with feedback

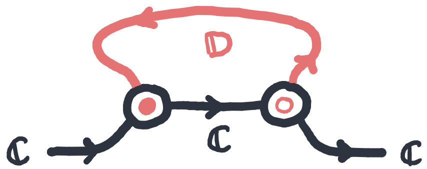
These are monads in the bicategory PROF of profunctors:

$$\hom(I, O) \qquad\qquad \int^{S \in C} \hom(S \otimes I, S \otimes O)$$

monads correspond to a new assignment of morphisms to a category.

Generalizing Feedback

What is the most general form of feedback?



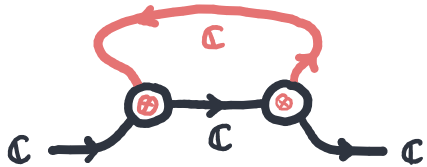
$$\mathrm{St}_{\mathbb{D}}(A,B) := \int^{D \in \mathbb{D}} \hom(D \bullet A,\ D \circ B)$$

The normal form theorem holds for any pair of monoidal actions $(\bullet,\circ)$. These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Generalizing Feedback

What is the most general form of feedback?



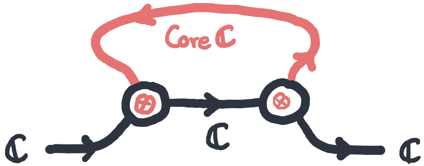
$$St_{\mathbb{C}}(A,B) := \int^{S \in \mathbb{C}} \hom(S \otimes A, S \otimes B)$$

The normal form theorem holds for any pair of monoidal actions $(\bullet, \circ)$. These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Generalizing Feedback

What is the most general form of feedback?



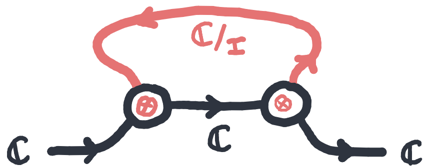
$$\mathrm{St}_{\mathrm{Core}\,\mathbb{C}}(A,B) := \int^{S \in \mathrm{Core}\,\mathbb{C}} \hom(S \otimes A, S \otimes B)$$

The normal form theorem holds for any pair of monoidal actions $(\bullet, \circ)$. These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Generalizing Feedback

What is the most general form of feedback?



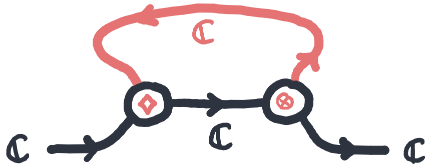
$$\mathrm{St}_{\mathbb{C}/_{I}}(A,B) := \int^{S,s \in \mathbb{C}/_{I}} \hom(S \circledast A, s \circledast B)$$

The normal form theorem holds for any pair of monoidal actions $(\bullet, \circ)$. These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Generalizing Feedback

What is the most general form of feedback?



$$St_{\mathbb{C},\diamond}(A,B) := \int^{S\in\mathbb{C}} \hom(\diamond S \otimes A, S \otimes B)$$

The normal form theorem holds for any pair of monoidal actions $(\bullet, \circ)$. These generalize:

- Right / Left traced categories.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Conclusion

- $\mathsf{Span}(\mathsf{Graph})_* \cong \mathsf{St}(\mathsf{Span}(\mathsf{Set}))$.
- $\mathsf{St}(\cdot)$ commonly appears across the literature.
- $\mathsf{St}(\cdot)$ is the free category with feedback.
- Categories with feedback are a weakening of traces.
- Categories with feedback have a normal form theorem.
- $\mathsf{St}(\cdot)$ can be generalized to variants of feedback.
- Relate to the coalgebraic approach.

References.

Katis, Sabadini, Walters. Feedback, trace, and fixed-point semantics.
RAIRO-Theoretical Informatics and Applications. Tome 36, 2002.

Katis, Sabadini, Walters. A categorial algebra of transition systems.
AMAST Proceedings. Ed: Michael Johnson. Springer, 1997.

Bonchi, Holland, Piedeleu, Sobociński, Zanasi. Diagrammatic algebra: from linear to concurrent systems.
Proc. ACM Program. Lang. (POPL), 2019.

Katsumata, Sprunger. Differentiable Causal Computations via Delayed Trace.
Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), 2019.

Hoshino, Muroya, Hasuo. Memoryful geometry of interaction.
Proc. Computer Science Logic (CSL), 2014.

Román. Open diagrams via Coend Calculus. (ACT'20).