Applied Category Theory 2021 ∘ The Gödel Fibration

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GÖDEL'S DIALECTICA INTERPRETATION

It is an interpretation of HA in the so-called system $\mathbb{T}$. Any formula A of HA is converted to the formula $A^D = (\exists x)(\forall y)A_D$, where A_D is quantifier-free, in such a way that we are as constructive as possible, while being able to interpret all of **classical arithmetic**.

CATEGORIFYING DIALECTICA INTERPRETATION

Hofstra [1] showed that Hyland and Biering's **Dialectica construction** $\mathfrak{Dial}$ associated to a fibration, which generalises de Paiva's notion of **Dialectica category** associated to a left exact category (the first attempt of internalising the Dialectica interpretation), can be seen as the composition of two free constructions $\mathfrak{Sum}$ and $\mathfrak{Prod}$, the *simple sum* and the *simple product* completions, which happen to be mutually dual.

The study of the monads $\mathfrak{Sum}$ and $\mathfrak{Prod}$ and their categorical properties constitutes the main part of our work and allows the proof of our main result (see [2] and arXiv:2104.14021 for more details).

OUR GOAL

To say when, for a given a fibration p, there is a fibration p′ such that $\mathfrak{Dial}(\mathrm{p}') \cong \mathrm{p}$ and, in this case, what p′ looks like.

BEING QUANTIFIER-FREE

Existential quantifier-free objects. For a given fibration $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ with simple coproducts, we say that an object $\alpha(i)$ over I is **$\coprod$-free** if it satisfies the following universal property:

- for every arrow $A \xrightarrow{f} I$ in B and every vertical arrow $\alpha(f(a)) \xrightarrow{h} (\exists b)\beta(a,b)$ over A, where the object $\beta(a,b)$ is over $A \times B$ for some B in B

there exist:

- a unique arrow $A \xrightarrow{g} B$ of B and a unique vertical arrow $\alpha(f(a)) \xrightarrow{\bar{h}} \beta(a, g(a))$ over A

such that h can be decomposed as
$\alpha(f(a)) \xrightarrow{\bar{h}} \beta(a,g(a)) \xrightarrow{\text{canonical}} (\exists b)\beta(a,b)$.

Analogously one can define the **universal quantifier-free objects** of p, i.e. the **$\prod$-free** object.

Gödel fibrations. If B is cartesian closed, a fibration $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ with simple products and simple coproducts is called a *Gödel fibration* if:

- the fibration p has enough $\coprod$-free objects *(i.e. any object in context $i : I$ is essentially of the form $(\exists a)\alpha(i,a)$ for some $\coprod$-free $\alpha(i,a)$)* which are stable under simple products and
- the sub-fibration $\bar{\mathrm{p}}$, whose elements are the $\coprod$-free objects, has enough $\prod$-free objects.

OUR RESULT

Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a fibration with simple products and simple coproducts and such that B is cartesian closed. Then there exists a fibration p′ such that $\mathfrak{Dial}(\mathrm{p}') \cong \mathrm{p}$ if and only if p is a Gödel fibration. In this case p′ is the sub-fibration of the $\prod$-free objects in the sub-fibration $\bar{\mathrm{p}}$ of $\coprod$-free objects of the fibration p.

FUTURE WORK

To generalise Hofstra's decomposition to the context of dependent type theory. To compare $\mathfrak{Dial}$ to other similar constructions in literature. To investigate applications to constructive mathematics and proof theory. To continue studying *preservation properties* of $\mathfrak{Sum}$, $\mathfrak{Prod}$ and $\mathfrak{Dial}$, as we do in arXiv:2104.14021.

REFERENCES

[1] Hofstra. The dialectica monad and its cousins. *Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai*, 53:107–139, 2011.

[2] Trotta, Spadetto, de Paiva. The Gödel fibration. In *46th International Symposium on Mathematical Foundations of Computer Science*, volume 171 of *LIPIcs*, 2021.