A Graphical Calculus for Lagrangian Relations

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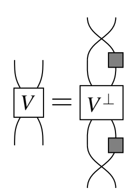
Symplectic vector spaces are the phase space of linear mechanical systems. The symplectic form describes, for example, the relation between position and momentum as well as current and voltage. The category of linear Lagrangian relations between symplectic vector spaces is a symmetric monoidal subcategory of relations which gives a semantics for the evolution – and more generally linear constraints on the evolution – of various physical systems.

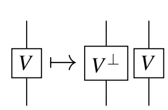
We give a new presentation of the category of Lagrangian relations over an arbitrary field as a 'doubled' category of linear relations. More precisely, we show that it arises as a variation of Selinger's CPM construction applied to linear relations, where the covariant orthogonal complement functor plays of the role of conjugation. Furthermore, for linear relations over prime fields, this corresponds exactly to the CPM construction for a suitable choice of dagger. We can furthermore extend this construction by a single affine shift operator to obtain a category of affine Lagrangian relations. Using this new presentation, we prove the equivalence of the prop of affine Lagrangian relations with the prop of qudit stabilizer theory in odd prime dimensions. We hence obtain a unified graphical language for several disparate process theories, including electrical circuits, Spekkens' toy theory, and odd-prime-dimensional stabilizer quantum circuits.

Linear Lagrangian relations, or more generally, affine Lagrangian relations provide a rich, compositional setting for modelling the evolutions of various physical systems. For example, certain classes of electrical circuits can be interpreted in terms of Lagrangian relations over the field of real rational functions [7, 8]. On a quite different note, the state preparation and quantum evolution of p-dimensional generalizations of Spekkens' toy theory [31] and (consequently) odd-prime-dimensional stabilizer quantum theory [22] have semantics in terms of affine Lagrangian relations over $\mathbb{F}_p$. Specifically, the state preparation corresponds to the affine Lagrangian relations from the tensor unit, and the evolution corresponds to affine symplectomorphisms. In this paper we extend this correspondance to the full category of Lagrangian relations, giving these circuits a proper categorical treatment.

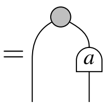
Formally, the category of Lagrangian relations is the symmetric monoidal subcategory of linear relations where the objects are symplectic vector spaces and the morphisms are linear relations satisfying an extra condition which can be captured graphically as the following, where $V^\perp$ denotes the orthogonal complement and the grey box denotes the *antipode* from the graphical theory of linear relations:



We show that any linear relation V determines a Lagrangian relation in terms of 'doubling', i.e. taking the tensor product of a linear relation with its complement:



By analogy to the CPM construction for the category of completely positive maps, we call these *pure* Lagrangian relations. In Theorem 3.2 we show that only one more class of 'discard' generators d_a for each a in the underlying field k is required to generate all Lagrangian relations.

$d_a :=$ 

From this, we immediately obtain a complete graphical calculus for Lagrangian relations over any field k, namely we can apply the complete calculus ih_k for linear relations [9] to diagrams built from pure morphisms and discard maps. This extends the doubled presentation of bond graphs, given in [18, 5.3], which are not universal for Lagrangian relations, and is instead only universal for a fragment of the pure morphisms. In Corollary 3.3, we also immediately get a *purification theorem* for Lagrangian relations, much like the purification (a.k.a. Stinespring dilation) of quantum channels which can be proven straightforwardly in the CPM construction over Hilbert spaces.

Furthermore, in the case of prime fields, i.e. finite fields $\mathbb{F}_p$ for p, we show in Corollary 3.4 that this is actually an instance of the original CPM construction, for a suitably defined dagger on the category of linear relations.

In Section 4 we show that only one more generator is needed to obtain *affine* Lagrangian relations. In the case of odd prime fields, we show in Theorem 4.16 that affine Lagrangian relations are prime-dimensional qudit stabilizer circuits, modulo invertible scalars. This give a graphical calculus extends to previous work on the qubit [4], and qutrit [32] cases. We also discuss the relation to electrical circuits.

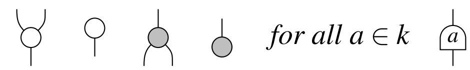
Related work. It was previously shown that certain classes of electrical circuits have a semantics in terms of affine Lagrangian relations over the field of the real numbers and the real rational functions $\mathbb{R}[x,y]/\langle xy-1\rangle$ [5, 7]. Similarly in [8, §VI], the authors give an interpretation of non-passive electrical circuits in terms of these 'doubled' string diagrams for affine relations over the real rational functions, however the authors did not give a full characterisation for the category of Lagrangian relations in terms of diagrammatic generators. We restate the interpretations of the electrical components given in [8, §VI] in terms of the graphical calculus for affine Lagrangian relations in Example 4.7.

A presentation of odd-prime stabilizer theory in terms of affine symplectomorphisms applied to Lagrangian subspaces appears in [22] and several follow-on works relating stabilizer theory to classical phase space via the discrete Wigner function. Our Theorem 4.16 is a categorical reformulation of the result of Spekkens' in which he shows that so called odd-prime-dimensional 'quadrature epirestricted theories' are operationally equivalent to prime-dimensional qudit stabilizer circuits [31]; following earlier work in [30]. This operational equivalence has also been further explored in the non-prime case [11]. Note that operational equivalence is not the same as categorical equivalence. The notion of operational equivalence used in [31, 11] refers to the equivalence of protocols in which circuits are prepared, evolve and then are measured; whereas ours is more 'process-theoretic', i.e. we consider the category that contains states, effects, evolutions, and all possible compositions thereof. A complete presentation for Spekkens' qubit toy model in terms of a category of relations has also been given [4] following the categorical description by [14]. However, the authors do not explicitly establish that this is the category of affine Lagrangian relations over $\mathbb{F}_2$, but merely a subcategory of finite sets and relations. There is also a complete presentation for qutrit stabilizer theory [32] which, by Theorem 4.16, is equivalent to Spekkens' qutrit toy model, up to scalars; the connection to relations, in this case, being unexplored.

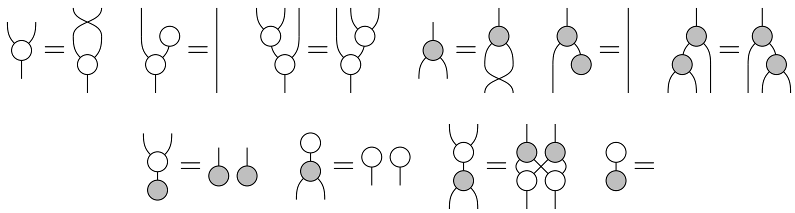
1 Linear relations

In order to decribe Lagrangian relations diagramatically, we must first recall the symmetric monoidal theory of linear relations. To do so, we first recall the symmetric monoidal theory of matrices:

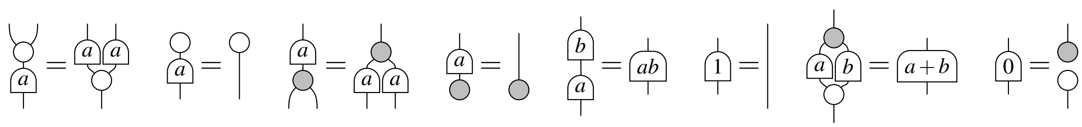
Definition 1.1. *[34, Defn. 3.4] Given a ring k, let* cb_k *denote the prop given by the generators*[1]*:*



modulo the equations of a bicommutative bialgebra:



and the additional equations:



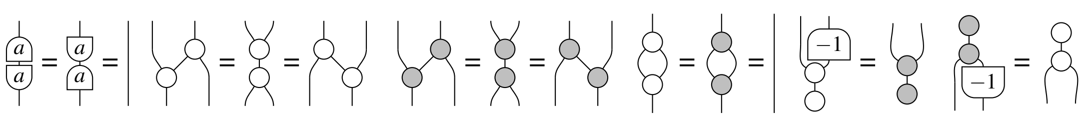
Proposition 1.2. *[34, Prop. 3.9] Given a ring k,* cb_k *is a presentation for the prop* Mat_k*, of matrices over k under the direct sum.*

One should interpret the grey monoid as addition and the white comonoid as copying.

Definition 1.3. *[34, Defn. 3.42] Given a field k, the prop of* **linear relations***,* $\mathsf{LinRel}_\mathsf{k}$*, has morphisms* $n \to m$ *as linear subspaces of* $k^n \oplus k^m$*, under relational composition and the direct sum as the tensor product.*

It is only necessary for k to be a principle ideal domain for composition to be well defined, but a field will do for the purposes of this paper.

Definition 1.4. *[34, Defn. 3.44] Given a field k, let* ih_k *denote the prop given by the quotient of the coproduct of props* $\mathsf{cb}_k^{\mathsf{op}} + \mathsf{cb}_k$ *by the following equations, for all invertible* $a \in k$ *(where the generators of* $\mathsf{cb}_k^{\mathsf{op}}$ *are drawn by reflecting those of* cb_k *along the x-axis):*



Theorem 1.5. *[34, Thm. 3.49]* ih_k *is a presentation for* $\mathsf{LinRel}_\mathsf{k}$*.*

There is an interesting folklore result which was remarked in [17] [2]:

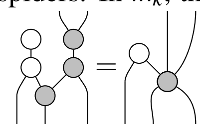
Lemma 1.6. *For a prime number p,* $\mathsf{ih}_{\mathbb{F}_p}$ *is a presentation for the phase-free, Fourier-free p-dimensional qudit ZX-calculus, modulo invertible scalars.*

[1] We use the ZX-style colouring which is dual to that used in [34].

[2] One should note that the black box is the antipode, *not* the Fourier transform/Hadamard gate.

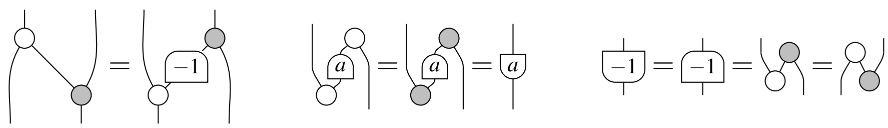
The following theorem will be useful for graphical manipulations:

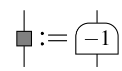
Theorem 1.7. *[2] (Spider Theorem) All connected components of special commutative Frobenius algebras with the same arity are equal.*

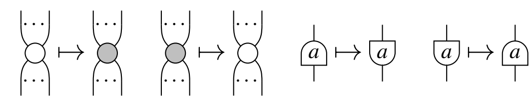
That is to say, we can unambiguously refer to these connected components by spiders. In ih_k, there are two spiders, so for example we can apply spider fusion to the following circuit: 

We shall use the following results:

Lemma 1.8. *[34, (D4)] [6, p. 4] [34, (D3)]*



Because of the symmetry of -1, we use the following (symmetric) notation for the antipode: 

Lemma 1.9. *[29] The functor* $(\_)^{\perp} : \mathsf{ih}_k \to \mathsf{ih}_k$; 

is the isomorphism which takes linear subspaces to their orthogonal complement, that is to say:

$$V \mapsto V^{\perp} := \{v \in V : \forall w \in V, \langle v, w \rangle = 0\}$$

Notice that the orthogonal complement is an involution so that $(V^{\perp})^{\perp} = V$.

2 Lagrangian relations

Now that we have a graphical presentation of linear relations, we can now do same for (linear) Lagrangian relations. We first recall some of the basic theory of symplectic vector spaces. This is expounded upon in much greater generality in the not-necessarily-linear case in [33]. In this entire paper, we only care about the linear and affine cases; and things will assumed to be linear unless otherwise stated. As previously mentioned, Lagrangian relations (and their affine counterpart) have previously been studied within the context of monoidal categories to model electrical circuits among other things [7, 5, 18]; although, to the knowledge of the authors, no proof of universality exists in the literature.

Definition 2.1. *Given a field k and a k-vector space V, a* **symplectic form** *on V is a bilinear map* $\omega : V \times V \to F$ *which is also:*

Alternating: *For all* $v \in V$, $\omega(v, v) = 0$.

Nondegenerate: *If* $\exists v \in V : \forall w \in V$ *we have* $\omega(v, w) = 0$*, then* $v = 0$.

A **symplectic vector space** *is a vector space equipped with a symplectic form.*

A (linear) **symplectomorphism** *is a linear isomorphism between symplectic vector spaces that preserves the symplectic form.*

Lemma 2.2. *Every vector space k^{2n} is equipped with a bilinear form given by the following block matrix:*

$$\omega := \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

so that $\omega(v,w) := v\omega w^T$. Moreover, every finite dimensional symplectic vector space over k is symplectomorphic to one of the form k^{2n} with such a symplectic form.

Definition 2.3. *Let $W \subseteq V$ be a linear subspace of a symplectic space V. The* **symplectic dual** *of the subspace W is defined to be $W^\omega := \{v \in V : \forall w \in W, \omega(v,w) = 0\}$. A linear subspace W of a symplectic vector space V is* **isotropic** *when $W^\omega \supseteq W$,* **coisotropic** *when $W^\omega \subseteq W$ and* **Lagrangian** *when $W^\omega = W$.*

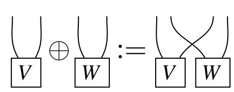
Lemma 2.4. *Every symplectomorphism $f : V \to V$ induces a Lagrangian relation $\Gamma_f := \{(fv,v)|v \in V\}$.*

These spaces have a natural grading into two distinct parts $V \oplus W \subseteq k^n \oplus k^n$. By analogy to the case of quantum stabilizer theory, we call the left part the *X-grading* and the right part the *Z-grading*.

As a matter of convention, we consider linear subspaces as being represented as the row space of a matrix. So in particular, a symplectic subspace of k^{2n} is represented by a matrix of the form $[X|Z]$ where X,Z are both $n \times n$-dimensional matrices. An isotropic subspace can equivalently be described as a matrix $[X|Z]$ so that $[X|Z]\omega[X|Z]^T = 0$. Moreover, a Lagrangian subspace can be described as a matrix as above which additionally has rank n.

Definition 2.5. *Given a field k, the prop of* **Lagrangian relations**, LagRel_k *has morphisms $k^{2n} \to k^{2m}$ as Lagrangian subspaces of the symplectic vector space $k^{n+m} \oplus k^{n+m}$ with symplectic form given above. Composition is given by relational composition and the tensor product is given by the direct sum.*

The direct sum of Lagrangian subspaces is graphically depicted as follows:



Where we are grouping the X gradings together on the left and the Z gradings together on the right. Note that this means the embedding of LagRel_k into $\mathsf{LinRel}_\mathsf{k}$ preserves the monoidal product only up to isomorphism. More precisely, we have the following fact.

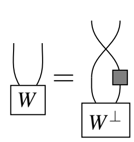
Lemma 2.6. *The forgetful functor $E : \mathsf{LagRel}_k \to \mathsf{LinRel}_\mathsf{k}$ is a faithful, strong symmetric monoidal.*

Proof. Functoriality and faithfulness is immediate. The strong monoidal structure is given by $E(I) = I$ and

$$E(A) \oplus E(B) := A \oplus A \oplus B \oplus B \xrightarrow{1\oplus\sigma\oplus 1} A \oplus B \oplus A \oplus B =: E(A \oplus B).$$

The symmetric monoidal structure on LagRel_k is chosen such that it is consistent with the monoidal structure above. □

Due to the above lemma, we will regard LagRel_k as a symmetric monoidal subcategory of $\mathsf{LinRel}_\mathsf{k}$. As such, we can ask what the generators of LagRel_k look like in terms of string diagrams of ih_k generators. We first describe what it means to be a Lagrangian relation in pictures, where the X block is the wire on the left and Z block is the wire on the right:

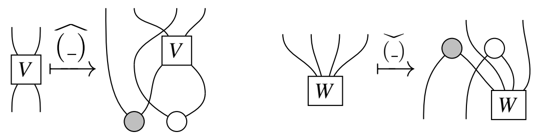


(1)

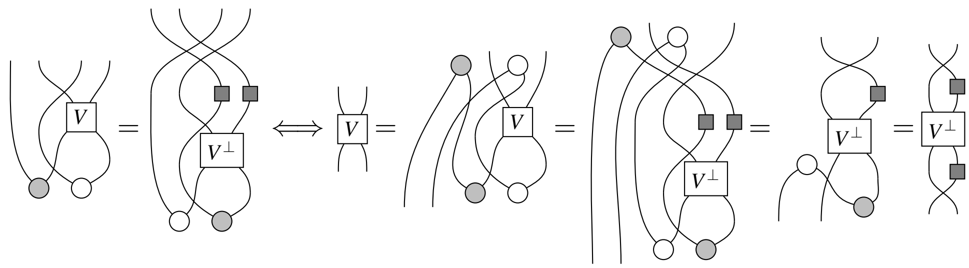
Algebraically, for W a subspace of V, the right hand side is interpreted as follows:

$$\begin{aligned} W^{\omega} := & \{(v_1,v_2) \in V : \forall (w_1,w_2) \in W, \omega((v_1,v_2),(w_1,w_2)) = 0\} \\ = & \{(v_1,v_2) \in V : \forall (w_1,w_2) \in W, \langle (v_2,-v_1),(w_1,w_2)\rangle = 0\} \\ = & \{(v_2,-v_1) \in V : \forall (w_1,w_2) \in W, \langle (v_1,v_2),(w_1,w_2)\rangle = 0\} \end{aligned}$$

The category of Lagrangian relations is compact closed. Given a relation V between symplectic vector spaces, we can curry it into a state $\widehat{V}$; and similarily, we can uncurry a state W into a process $\check{W}$,

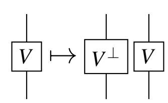


It is easy to see that these two constructions are inverse to each other. This allows us to derive a graphical criteria for abitrary Lagrangian relations, generalizing Equation 1:

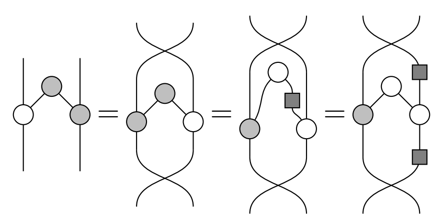


For this reason, we will depict Lagrangian relations as processes, where the input wires are on the bottom and output wires on on the top.

Lemma 2.7. *There is a faithful, strong symmetric monoidal functor $L : \mathsf{LinRel_k} \to \mathsf{LagRel_k}$ given by the following action on the generators of ih_k; doubling, and then changing the colours of one of the copies:*



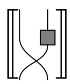
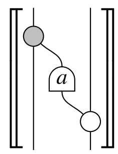
To check this is a functor, all we have to show is that it produces Lagrangian relations. This follows immediately from the naturality of -1. This functor is symmetric monoidal and faithful but not full, as for example, the following Lagrangian relation is not in the image of L:

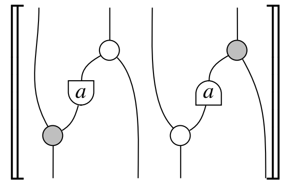


3 Generators for Lagrangian relations

In this section, we shall give a universal set of generators for LagRel_k; although, we do not directly give a complete set of identities. Instead we defer to the completeness of the underlying category $\mathsf{ih}_k \cong \mathsf{LinRel_k}$.

Consider the following symplectomorphisms; the discrete Fourier transform F, the a-shift gate S_a and the controlled-a gate C_a:


$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$


$= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$


$= \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \end{bmatrix}$

Use the notation $G^{(j)}$ to denote a gate G being applied to wire j; and the notation $C_a^{(i,j)}$ to denote the controlled-a gate controlling on wire i targetting wire j.

Note the right action of these gates in terms of matrix multiplication of Lagrangian subspaces for any nonzero $a \in k$ (as observed in [1, p. 4]):

- $F^{(i)}$ sets columns x_i to $-z_i$ and z_i to x_i.
- $S_a^{(i)}$ sets z_i to $z_i + a \cdot x_i$.
- $C_a^{(i,j)}$ sets x_j to $x_j - a \cdot x_i$ and z_i to $z_i + a \cdot z_j$.

Using these symplectomorphisms regarded as Lagrangian relations, we have:

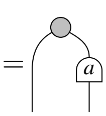
Theorem 3.1. *For any field k the maps in $L(\mathsf{LinRel_k})$ as well as F and S_a for all $a \in k$ generate LagRel_k.*

See Appendix A for the proof. The idea is that we use these symplectomorphisms to reduce a Lagrangian relation to the state $L(\circ^{\otimes n})$.

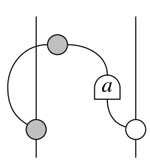
We can also give a presentation of this category which is very close to Selinger's CPM construction [28]. There are several equivalent ways to define the CPM construction. For our purposes, the most convenient one is the presentation used in both in [12, 16], which defines $\mathsf{CPM}[\mathbb{X}]$ as the subcategory of a dagger compact closed category $\mathbb{X}$ whose objects are of the form $A^* \otimes A$ for $A \in \mathbb{X}$ and whose morphisms are generated by (i) 'pure' morphisms, i.e. morphisms of the form $f_* \otimes f$ for $f \in \mathbb{X}$ and a covariant functor $(\_)_*$, and (ii) a 'discard' morphism d_A for every $A \in \mathbb{X}$ given by the counit $d_A := \varepsilon_A : A^* \otimes A \to I$ of the compact closed structure on A.

We nearly obtain such a presentation for LagRel_k using the covariant functor $(-)^\perp$ to define pure morphisms, with the only caveat being we need to consider a family of discard morphisms: each discard morphism being parametrised by a field element.

Theorem 3.2. *LagRel_k is the monoidal subcategory of $\mathsf{LinRel_k}$ whose objects are of the form $2^n \oplus 2^n$, for all natural numbers n, and whose morphisms are generated by pure morphisms of the form $V^\perp \oplus V$ for $V \in \mathsf{LinRel_k}$ and for each $a \in k$, a 'discard' morphism:*

$d_a :=$


Proof. We just have to show that F and S_a can be constructed using these generators. The S_a gate and it's colour-reversed version V_a can be obtained by composing a pure morphism with d_a and d_{-a}, respectively:



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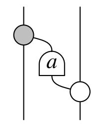
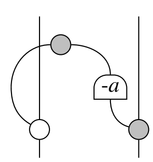


$= S_a$


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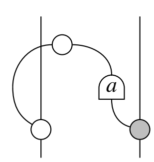
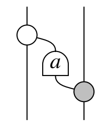
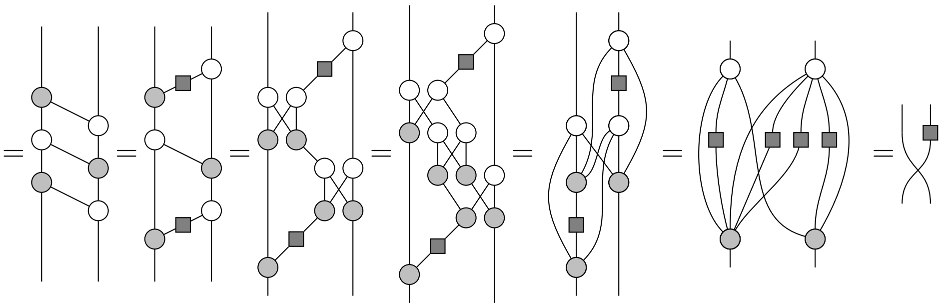


$=$


$=: V_a$

We can then obtain F as $S_1 \circ V_1 \circ S_1$, which can be proven as a variation of the familiar '3 CNOT' rule for quantum circuits (see e.g. [13, §3.2.1]):

$S_1 \circ V_1 \circ S_1 =$



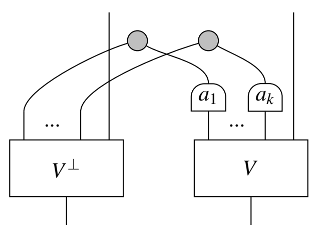
$= F$

□

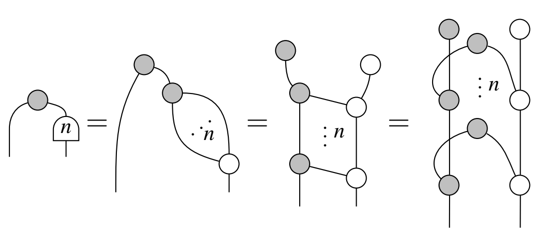
In the ZX-calculus literature, this decomposition of the Fourier transform is known as *Euler decomposition* [20]. A variant of this decomposition is given in [6, p.6]; although in the context of plain old linear relations instead of Lagrangian relations, so an antipode is missing in their case. A similar observation was made in [26, (34)] in terms of qudit controlled boost gates; however, the connection to phase-shift gates and Euler decomposition was not made.

From Theorem 3.1, we know that we can build any Lagrangian relation using pure Lagrangian relations and discard maps. Since the former is closed under composition and monoidal product, the following can be shown immediately from string diagram deformation.

Corollary 3.3 (Purification)**.** *Any linear Lagrangian relation can be written in the following form, for V a linear relation:*



In the case when we are working with prime fields, then Lagrangian relations are exactly an instance of the CPM construction. Namely, in the category of linear relations, $(-)^*$ is given by relational converse, so we can define a dagger functor $(-)^\dagger := ((-)^\perp)^*$ such that $(-)_* = (-)^\perp$. It only remains to show that all of the discarding maps arise from a single fixed cap. This can be done as follows, for $k = F_p$:

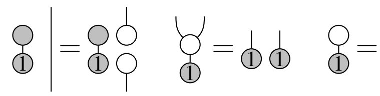


Corollary 3.4. *For p prime,* $\mathsf{LagRel}_{\mathbb{F}_p} \cong \mathsf{CPM}[\mathsf{LinRel}_{\mathbb{F}_p}]$.

4 Affine Lagrangian relations

Affine Lagrangian relations are perhaps of more practical interest than plain old Lagrangian relations. As we will discuss in this section, these give a semantics for qudit stabilizer circuits as well as certain electrical circuits. We use our universal set of generators for Lagrangian relations as well as the presentation for affine relations to get a universal set of generators for affine Lagrangian relations.

Definition 4.1. *[8, §A] Let* aih_k *denote the the prop presented by* ih_k *in addition to the generator* $\textcircled{1}$ *and three equations:*

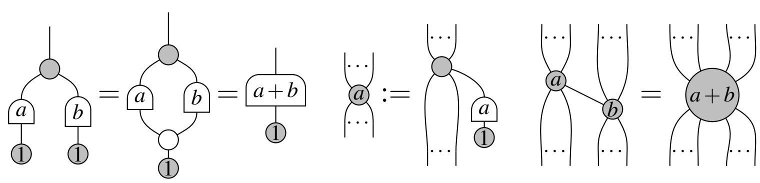


The following was stated slightly differently in the original paper:

Definition 4.2. *[8, Definition 5] Let* AffRel_k *denote prop, whose morphisms* $n \to m$ *are the (possibly empty) affine subspaces of* $2^n \oplus 2^m$*; with composition given by relational composition and tensor product given by the direct sum.*

Theorem 4.3. *[8, Thm. 17]* aih_k *is a presentation of* AffRel_k.

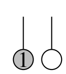
Because the equation on the left holds, we can use the phased-spider notation (as in the ZX-calculus), so that for all $a, b \in \mathbb{F}_p$:



Definition 4.4. *Let* $\mathsf{AffLagRel}_k$ *denote the monoidal category whose objects are symplectic vector spaces, and whose morphisms are generated by the image of* $\mathsf{LagRel}_k \xrightarrow{E} \mathsf{LinRel}_\mathsf{k} \to \mathsf{AffRel}_\mathsf{k}$ *as well as all affine shifts and whose tensor product is the direct sum.*

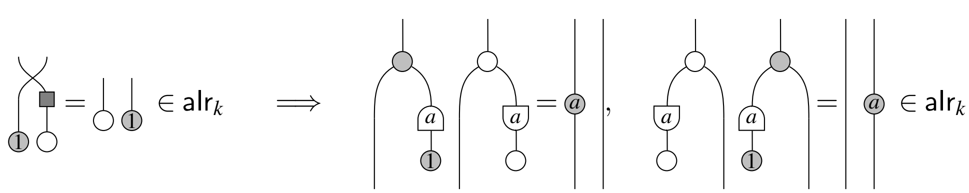
Because the tensor product is defined in the same way as in LagRel_k, as in Lemma 2.6, the forgetful functor $\mathsf{AffLagRel}_k \to \mathsf{AffRel}_k$ is faithful, but only *strong* monoidal.

Definition 4.5. *Let* alr_k *denote the monoidal subcategory of* aih_k *with objects* $2n$*, generated by the morphisms in the image of* $\mathsf{LagRel}_k \xrightarrow{E} \mathsf{LinRel}_\mathsf{k} \cong \mathsf{ih}_\mathsf{k} \to \mathsf{aih}_\mathsf{k}$ *as well as the following generator:*



Lemma 4.6. alr_k *is a presentation of* $\mathsf{AffLagRel}_k$.

Proof. All the affine shifts can be produced from tensoring and composing these two maps on the right:

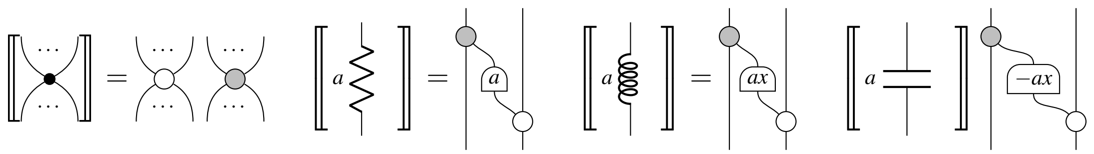


□

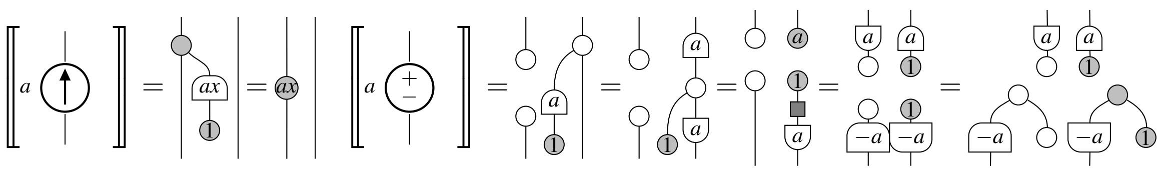
Therefore, we are justified in using string diagrams in alr_k to reason about morphisms in $\mathsf{AffLagRel}_k$.

We will restate the interpretations given in [8] of some components for electrical circuits in terms affine relations in terms of the generators for graphical calculus for Lagrangian relations. This interpretation is also explored in [7, 5]; albeit, not enjoying the graphical calculus for affine relations.

Example 4.7. *For any non-negative real a, wires, a-weighted resistors, inducers and capacitors have the following interpretations in* $\mathsf{AffLagRel}_{\mathbb{R}[x,y]/\langle xy-1\rangle}$*:*



Similarly for a-valued voltage and current sources (again, for a a non-negative real number):



Note that these generators do not generate the whole category of Lagrangian relations; for instance, the coefficients are required to be non-negative.

4.1 Stabilizer circuits and Spekkens' toy model

In this subsection, we show that, when p is an odd prime, the prop of affine Lagrangian relations over $\mathbb{F}_p$ is isomoprhic to p-dimensional qudit stabilizer circuits, modulo invertible scalars. We first consider an intermediary fragment between the Fourier-free, phase free fragments and stabilizer circuits.

Definition 4.8. *The qudit* **boost operator** *is the following unitary on d in* $\mathsf{Mat}(\mathbb{C})$, $\mathscr{X} := \sum_{a=0}^{d-1} |a+1\rangle\langle a|$.

In the qubit case, the boost operator is just the not gate. Adding the affine shift to $\mathsf{ih}_{\mathbb{F}_p}$, corresponds to adding the boost gate to the Fourier-free, phase-free ZX-calculus, extending Lemma 1.6. This is a qudit generalization of the observation made in [17]:

Lemma 4.9. *For p prime,* $\mathsf{aih}_{\mathbb{F}_p}$ *is isomorphic as a prop to the Fourier-free, p-dimensional qudit ZX-calculus with the boost operator modulo invertible scalars.*

We can go further with affine Lagrangian relations. Inspired by the work of Spekkens [30, 31]:

Definition 4.10. *When p is prime, let* **Spekkens' qudit toy model** *of dimension p denote the prop* $\mathsf{AffLagRel}_{\mathbb{F}_p}$.

We first give a short review of the qudit stabilizer formalism, before establishing the equivalence between Spekkens' toy model and stabilizer circuits in the odd prime qudit case. All of the material from Definition 4.11 to 4.13 are contained in [24].

Definition 4.11. *The qudit* **shift operator** *is the following unitary on d in* $\mathsf{Mat}(\mathbb{C})$, $\mathscr{Z} := \sum_{a=0}^{d-1} e^{2\pi i a/d}|a\rangle\langle a|$.

The n-qudit **Pauli group** $\mathfrak{P}_d^{\otimes n}$ *is defined to be the subgroup of* $\mathsf{Mat}(\mathbb{C})(d^n, d^n)$ *generated by the shift and boost operators as well as* I_d *and the scalar* $e^{\pi i/d}$ *under tensor product and matrix multiplication.*

An n-qudit **Clifford operator** *U is an d^n-dimensional unitary so that $U\mathfrak{P}_d^n U^\dagger = \mathfrak{P}_d^n$.*

The n-qudit **Clifford group** *is formed by the n-qudit Clifford operators under matrix multiplication.*

An n-qudit **stabilizer state** *is a state $U|0\rangle^{\otimes n}$ for an n-qudit Clifford U.*

Given any n-qudit stabilizer state $|\psi\rangle$, the **stabilizer group** *of $|\psi\rangle$ is the (Abelian) subgroup of $\mathfrak{S}_{|\psi\rangle} \subset \mathfrak{P}_d^n$ whose elements are the $U \in \mathfrak{P}_d^n$ for which $U|\psi\rangle = |\psi\rangle$.*

Lemma 4.12. *Two stabilizer states with the same stabilizer groups are the same, up to global phases.*

Lemma 4.13. *For natural numbers $n, d \geq 2$ the n-dimensional qudit stabilizer group modulo invertible scalars is generated under tensor and composition of I_d as well as the boost operator $\mathscr{X}$, the controlled-boost operator $\mathscr{C}$, the Fourier transform $\mathscr{F}$ and the phase-shift operator $\mathscr{S}$:*

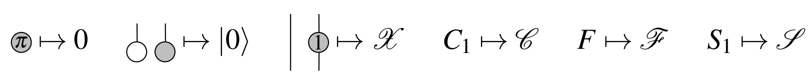
$$\mathscr{C} := \sum_{a,b=0}^{d-1} |a, a+b\rangle\langle a, b| \qquad \mathscr{F} := \frac{1}{\sqrt{d}} \sum_{a,b=0}^{d-1} e^{2\pi i ab/d} |b\rangle\langle a| \qquad \mathscr{S} := \sum_{a=0}^{d-1} e^{\pi i a(a+d)/d} |a\rangle\langle a|$$

Notice that the boost operator can be obtained by $\mathscr{Z} = \mathscr{F}\mathscr{X}\mathscr{F}^2$.

Definition 4.14. *Let Stab_p denote the subcategory of $\mathsf{Mat}(\mathbb{C})$ generated by the p-dimensional qudit Clifford group as well as the vectors $|0\rangle$, $\langle 0|$, quotiented by invertible scalars.*

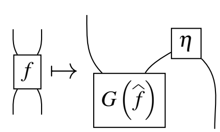
The following isomorphism is described in [22], when restricted to the nonempty case. This comes from the projective representation of the n qudit odd-prime-dimensional Clifford group in terms of the affine symplectomorphisms over $\mathbb{F}_p^n$. However, since there is only one empty relation and one zero matrix of every type, we get the following result immediately:

Lemma 4.15. *For every odd prime p, there is an isomorphism $G : \mathsf{AffLagRel}_{\mathbb{F}_p}(\mathbb{F}_p^0, \mathbb{F}_p^{2n}) \to \mathsf{Stab}_p(0, n)$ determined by:*



We extend this isomorphim of states to an isomorphism of props:

Theorem 4.16. *When p is an odd prime, the mapping $H : \mathsf{AffLagRel}_{\mathbb{F}_p} \to \mathsf{Stab}_p$ defined by:*



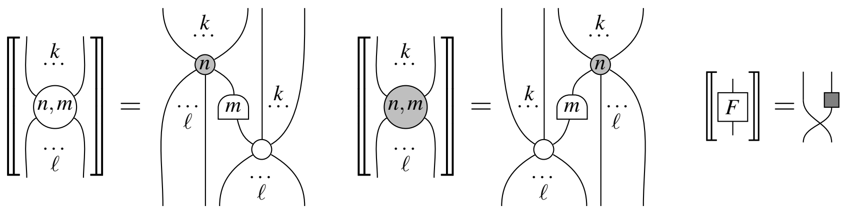
is a symmetric monoidal equivalence, where η is the cap of the compact closed structure induced by the Z observable.

See Appendix B for the proof. The main difficulty is showing that H is a functor, but the idea is to use the fact that stabilizer states with the same stabilizer group only differ by a global scalar.

As we mentioned in the intro, this is a categorical reformulation of the result of Spekkens' in which he shows that odd-prime-dimensional 'quadrature epirestricted theories' are operationally equivalent to prime-dimensional qudit stabilizer circuits [31].

A complete presentation for Spekkens' qubit toy model in terms of a category of relations was given [4] in a style which mirrors that of the qubit ZX-calculus [13]. We now show how the generators of that presentation appear in our 'doubled' formulation.

Remark 4.17. *We can present Spekkens' p-dimensional qudit toy model in a manner similar to the ZX-calculus, in terms of being generated by spiders with phases labelled by the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and for arbitrary natural numbers k,ℓ:*



The Fourier transform is redundant, as it can be obtained by Euler decomposition.

Notice that the phases of the Z and X observables are elements (n,m) of $\mathbb{F}_p \times \mathbb{F}_p$, and it is easy to see how the doubled spiders satisfy the phased spider fusion laws with respect to the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, as discussed in [27, p.166]. As discussed in [15] this is one of the central features which separates Spekkens' qubit model from qubit stabilizers, whose phase group is $\mathbb{Z}/4\mathbb{Z}$. This fact can be also observed graphically in terms of the stabilizer fragment of the ZX-calculus (in contrast to the presentation of Spekkens' qubit toy model) which also enjoys a complete axiomatization [4].

One should also note that this idea of representing stabilizer circuits in terms of 'doubling' quantum circuits was also used by [25].

By stating the interpretations of Spekkens' toy model in terms of the graphical calculus for Lagrangian relations alongside that of electrical circuits, we see the evident analogy between the phases in the ZX-calculus and the resistors, inducers, capacitors and voltage sources in electrical circuits.

5 Further work

There are several directions in which the work in this paper could be further explored. Since linear relations are can be defined over a principle ideal domain over a field, it is natural to ask if the work can be generalized to this setting.

Also, we have not given a proper completeness result. The proof of such would almost certainly involve mimicking the universality proofs of the qubit stabilizer/qutrit stabilizer/Spekkens' toy model [3, 4, 32] involving local equivalency/local complementation of graph states. If this were generalized to affine Lagrangian relations this would yield a proper completeness result for the odd-prime-dimensional qudit stabilizer ZX calculus as a corollary.

One could also potentially adapt this approach to characterize Lagrangian spans as described in [21, p. 187], where the scalars are not all quotiented out. Perhaps this would give a semantics for odd-prime-dimensional qudit stabilizer circuits on the nose.

This paper illuminates the deep connection between stabilizer circuits and electrical circuits. Perhaps, this can be taken further by adding nonlinear generators. For example, by doubling again, one could also consider discarding as a generator, as in [10]. Does the discard in stabilizer quantum mechanics obey analagous equations to the ground in electrical circuits?

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A Proof of Theorem 3.1

Theorem 3.1 *For any field k the maps in $L(\mathsf{LinRel}_\mathsf{k})$ as well as F and S_a for all $a \in k$ generate* LagRel_k.

Proof. Consider the matrix $[X|Z]$ of an arbitrary Lagrangian relation over field k seen as a state. We show how one can reduce $[X|Z]$ to the block matrix $[I|0]$ by right multiplication with the aforementioned symplectomorphisms. To do so, we first reduce it to a matrix $[I|Z']$ by only applying row operations (keeping the subspace the same) as well as the Fourier transform. This involve repeatedly do Gaussian elimination and then applying the Fourier transform to wires when the pivot is in the Z block. We are guaranteed to obtain a matrix $[I|Z']$ because the dimension of Lagrangian subspace is n. A very similar observation was made in [1, Lem. 6].

As the Fourier transform is a symplectomorphism $[I|Z']$ is isotropic, so that:

$$0 = \left[I|Z'\right] \omega \left[I|Z'\right]^T$$

which holds if and only if

$$0 = \left[I|Z'\right] \left[Z'|-I\right]^T = Z'^T - Z'$$

That is to say Z' is symmetric, meaning that Z' describes the adjacency matrix of a graph coloured by the elements of k. In the language of stabilizer circuits, this is called a *graph state*. In the case of prime fields, this observation was made in [22, Eq. 18]. Graph states were originally discussed in [23].

We prove that graph states can be reduced to the subspace $[I|0]$ by induction on the dimension of the subspace. This base case is trivial.

Suppose we have a $(n+1)$-dimensional Lagrangian subspaces described by a graph state, then:

$$
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \cdots & 0 & z_{1,1} & z_{1,2} & z_{1,3} & \cdots & z_{1,n} \\
0 & 1 & 0 & \cdots & 0 & z_{1,2} & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
0 & 0 & 1 & \ddots & \vdots & z_{1,3} & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & z_{1,n} & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
\xmapsto{(F^{(1)})^{-1}}
\left[\begin{array}{ccccc|ccccc}
z_{1,1} & 0 & 0 & \cdots & 0 & -1 & z_{1,2} & z_{1,3} & \cdots & z_{1,n} \\
z_{1,2} & 1 & 0 & \cdots & 0 & 0 & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
z_{1,3} & 0 & 1 & \ddots & \vdots & 0 & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{1,n} & 0 & \cdots & 0 & 1 & 0 & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
$$

$$
\xmapsto{C^{(2,1)}_{z_{1,2}}}
$$

$$
\left[\begin{array}{ccccc|ccccc}
z_{1,1}-0 & 0 & 0 & \cdots & 0 & -1 & z_{1,2}-z_{1,2} & z_{1,3} & \cdots & z_{1,n} \\
z_{1,2}-z_{1,2} & 1 & 0 & \cdots & 0 & 0 & z_{2,2}-0 & z_{2,3} & \cdots & z_{2,n} \\
z_{1,3}-0 & 0 & 1 & \ddots & \vdots & 0 & z_{2,3}-0 & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{1,n}-0 & 0 & \cdots & 0 & 1 & 0 & z_{2,n}-0 & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
=
\left[\begin{array}{ccccc|ccccc}
z_{1,1} & 0 & 0 & \cdots & 0 & -1 & 0 & z_{1,3} & \cdots & z_{1,n} \\
0 & 1 & 0 & \cdots & 0 & 0 & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
z_{1,3} & 0 & 1 & \ddots & \vdots & 0 & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{1,n} & 0 & \cdots & 0 & 1 & 0 & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
$$

$$
\xmapsto{\prod_{i>1}^{n} C^{(i,1)}_{z_{1,i}}}
$$

$$
\left[\begin{array}{ccccc|ccccc}
z_{1,1} & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
0 & 0 & 1 & \ddots & \vdots & 0 & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
\xmapsto{F^{(1)}}
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \cdots & 0 & z_{1,1} & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
0 & 0 & 1 & \ddots & \vdots & 0 & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
$$

$$
\xmapsto{S^{(1)}_{-z_{1,1}}}
$$

$$
\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & z_{2,2} & z_{2,3} & \cdots & z_{2,n} \\
0 & 0 & 1 & \ddots & \vdots & 0 & z_{2,3} & z_{3,3} & \cdots & z_{3,n} \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & z_{2,n} & z_{3,n} & \cdots & z_{n,n}
\end{array}\right]
$$

Therefore all Lagrangian relations can be reduced to the subspace $[I|0]$ by right multiplication by symplectomorphisms. In the n-dimensional case, this subspace is given by the circuit $L(⫰^{\otimes n})$.

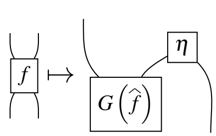
Thus, we have already described all the generators of Lagrangian relations. The gates F, C_a for all $a \in k$, along with the cup and cap and the zero state generate all Lagrangian relations. Note that C_a and the zero state are both in the image of the L.

□

B Proof of Theorem 4.16

Recall the statement of the theorem:

Theorem 4.16 *When p is an odd prime, the mapping $H : \mathsf{AffLagRel}_{\mathbb{F}_p} \to \mathsf{Stab}_p$ defined by:*

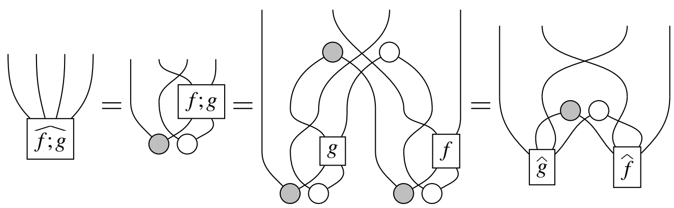


is a symmetric monoidal equivalence, where η is the cap of the compact closed structure induced by the Z observable.

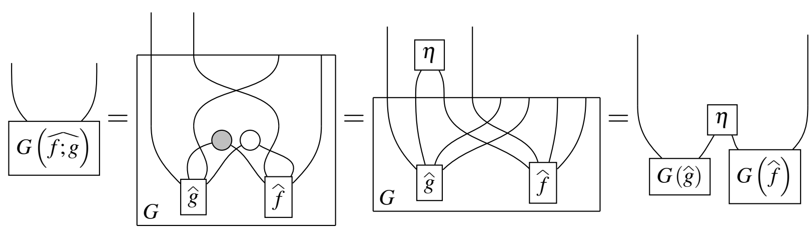
Proof. It preserves identities by the snake equations. Now we must show it preserves composition. Consider some composable pair in $\mathsf{AffLagRel}_{\mathbb{F}_p}$:

$$\mathbb{F}_p^{2n} \xrightarrow{f} \mathbb{F}_p^{2m} \xrightarrow{g} \mathbb{F}_p^{2\ell}$$

If the composite is empty, then the result follows immediately. Suppose otherwise. We know that:

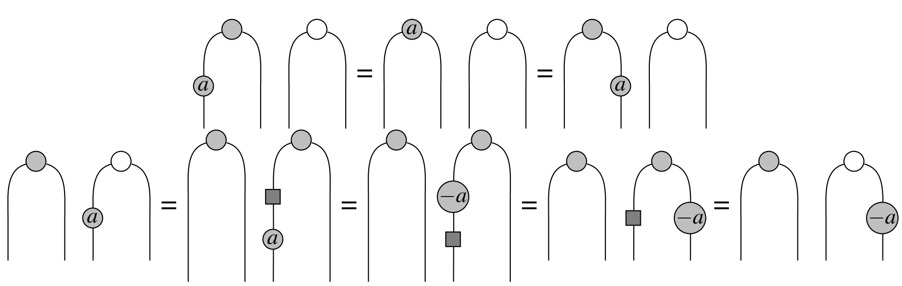


We have the following equality of diagrams in Stab_p, We draw the wires exiting G to be connected to the corresponding wires in the X block of the subspace.

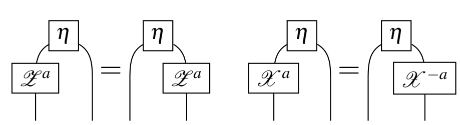


This second equality is the only nontrivial part. It follows by observing that both stabilizer states are stabilized by the same generalized Pauli operators, and thus they are the same. This is because the generalized Pauli operators can be pulled through G, by [22, Lemma 4], where they act the same on the caps of Lagrangian relations and in matrices.

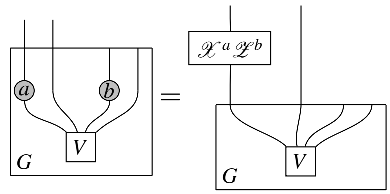
Explicity, the boost and shift operators commute with the cap as follows:



Which are analagous to the following commutations in stabilizer circuits:



Where moreover, for any Lagrangian relation V and $n,m \in \mathbb{F}_p$, we already know:



Therefore, functoriality follows by uncurrying the left and right hand sides of the previous equation. Fullness and faithfulness follow immediately from G being an isomorphism. □