Lovász-Type Theorems and Game Comonads
(extended abstract)

Anuj Dawar Tomáš Jakl
Department of Computer Science and Technology,
University of Cambridge, UK
anuj.dawar@cl.cam.ac.uk, tj330@cam.ac.uk

Luca Reggio
Department of Computer Science
University of Oxford, UK
luca.reggio@cs.ox.ac.uk

Lovász (1967) showed that two finite relational structures A and B are isomorphic if, and only if, the number of homomorphisms from C to A is the same as the number of homomorphisms from C to B for any finite structure C. Soon after, Pultr (1973) proved a categorical generalisation of this fact. We propose a new categorical formulation, which applies to any locally finite category with pushouts and a proper factorisation system. As special cases of this general theorem, we obtain two variants of Lovász' theorem: the result by Dvořák (2010) and the result of Grohe (2020). They both characterise the indistinguishability of graphs with respect to a fragment of first-order logic with counting quantifiers in terms of homomorphism counts from graphs of tree-width (resp. tree-depth) at most k. The connection of our categorical formulation with these results is obtained by means of the game comonads of Abramsky et al. (2017, 2018) We also present a novel application to homomorphism counts in modal logic.

1 Background

Over fifty years ago, Lovász [10] proved that two finite relational structures A and B are isomorphic if, and only if, the number of homomorphisms from C to A is the same as the number of homomorphisms from C to B, for any finite structure C. Not long after Pultr [12] proved a categorical generalisation of this fact. He showed that every finitely well-powered, locally finite category $\mathscr{A}$ with (extremal epi, mono) factorisation is *combinatorial*, that is, for $a, b \in \mathscr{A}$,

$$a \cong b \quad \iff \quad |\hom_{\mathscr{A}}(c,a)| = |\hom_{\mathscr{A}}(c,b)| \qquad \text{for every } c \text{ in } \mathscr{A}$$

where $|\hom_{\mathscr{A}}(c,a)|$ denotes the number of morphisms $c \to a$ in $\mathscr{A}$. Similar categorical generalisations with slightly different assumptions were also proved by Isbell [8] and Lovász [11]. Note that the difference from Yoneda Lemma is that naturality in c is not required.

We provide a new categorical generalisation of Lovász' theorem:

Theorem 1. *Let $\mathscr{A}$ be a locally finite category. If $\mathscr{A}$ has pushouts and a proper factorisation system $(\mathscr{E}, \mathscr{M})$, then it is combinatorial.*

By a *proper factorisation system* we mean a weak factorisation system $(\mathscr{E}, \mathscr{M})$ such that $\mathscr{E}$ is a class of epimorphisms and $\mathscr{M}$ is a class of monomorphisms. It is immediate to see that Lovász' theorem follows from Theorem 1, when $\mathscr{A}$ is taken to be the category Σ_f of finite σ-structures with homomorphisms, for a fixed finitary relational signature σ.

[0]Research supported in part by the EPSRC grant EP/T007257/1, and by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 837724.

The full text to appear in the Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)

Submitted to:
ACT 2021

What sets our result apart from the other categorical generalisations is that our proof uses elementary facts about *polyadic spaces* (cf. Joyal [9]), which are the Stone duals of Boolean hyperdoctrines, in order to show that the unnatural isomorphism of $\hom_{\mathscr{A}}(-,a)$ and $\hom_{\mathscr{A}}(-,b)$ implies an unnatural isomorphism of $\mathscr{M}(-,a)$ and $\mathscr{M}(-,b)$, where $\mathscr{M}(c,a)$ is the set of $\mathscr{M}$-morphisms $c \to a$. Moreover, the usual combinatorial counting argument is eliminated by referring to the Principle of Inclusion and Exclusion. Theorem 1 is also well-suited for applications to game comonads that we discuss next.

1.1 Refinements

The seminal result of Lovász [10] has led to extensions and investigations in many different directions. Notably for us, Grohe [7] recently proved the following refinement: for finite graphs A,B,

$$A \equiv_{\mathscr{C}_n} B \quad\iff\quad |\hom(C,A)| = |\hom(C,B)|$$

for every finite graph C of *tree-depth* $\leq n$. Where $A \equiv_{\mathscr{C}_n} B$ denotes that A and B are indistinguishable in $\mathscr{C}_n$, that is, in the first-order logic with counting quantifiers and quantifier depth at most $\leq n$. Similarly, Dvořák [6] showed that homomorphism counts from graphs of *tree-width* $\leq k$ classify graphs up-to indistinguishability in $\mathscr{C}^{k+1}$, which is the first-order logic with counting quantifiers and restricted to no more than $k+1$ distinct variables [4].

An interplay between fragments of first-order logic and combinatorial parameters (such as tree-width and tree-depth) is also typical for game comonads of Abramsky et al. [1, 3]. We show that these game comonads provide the connection between Theorem 1 and the results of Dvořák and Grohe.

1.2 Game comonads

The *Ehrenfeucht–Fraïssé comonad* $\mathbb{E}_n$, for a fixed positive integer n (cf. [3]), is a comonad on the category Σ of σ-structures, where σ is a relational signature. The elements of $\mathbb{E}_n(A)$, for a σ-structure A, are representing states in the Ehrenfeucht–Fraïssé game, namely $\mathbb{E}_n(A)$ consists of non-empty sequences of length $\leq n$.

As shown already in [3], the Ehrenfeucht–Fraïssé comonad can express both the logical and combinatorial properties mentioned in Grohe's theorem. Namely, tree-depth of a σ-structure A is expressible by admitting a coalgebra structure:

$$\exists \text{ a comonad coalgebra } A \to \mathbb{E}_n(A) \quad\iff\quad A \text{ has tree-depth at most } n.$$

Similarly, the indistinguishability by the $\mathscr{C}_n$ fragment of logic is expressed by an isomorphism of cofree coalgebras. To this end, set σ^+ to be the relational signature σ extended with a fresh binary relation I. Extending the signature is necessary in order to capture equality in the logic. As before we have a category Σ^+ of σ^+-structures and the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_n^+$ on Σ^+. Then, for σ-structures A,B,

$$A \equiv_{\mathscr{C}_n} B \quad\iff\quad F^{\mathbb{E}_n^+}(J(A)) \cong F^{\mathbb{E}_n^+}(J(B))$$

where $F^{\mathbb{E}_n^+} : \Sigma^+ \to \mathrm{EM}(\mathbb{E}_n^+)$ assigns to the σ^+-structure A the cofree Eilenberg–Moore coalgebra $\mathbb{E}_n^+(A) \to \mathbb{E}_n^+(\mathbb{E}_n^+(A))$, and

$$J \colon \Sigma \to \Sigma^+$$

is the functor that maps a σ-structure A to the σ^+-structure $J(A)$, obtained from A by interpreting I as the equality symbol. Note that J has a left adjoint $H \colon \Sigma^+ \to \Sigma$, mapping a σ^+-structure A to the σ-structure reduct of A quotiented by the transitive, symmetric and reflective closure of the added relation I.

Similarly, the indistinguishability by the $\mathscr{C}^k$ fragment and tree-width, appearing in Dvořák's theorem, can be expressed in terms of the *pebbling comonad* $\mathbb{P}_k$ [1].

2 Categorical proofs of Dvořák's and Grohe's theorems

Both Grohe's and Dvořák's theorems can be expressed in terms of the Ehrenfeucht–Fraïssé and pebbling comonads, respectively. We devised a general categorical framework that is parametric in the comonad $\mathbb{C}$ on Σ and from which the two theorems follow. To this end, for the forgetful functor $U^{\mathbb{C}}\colon \mathrm{EM}(\mathbb{C}) \to \Sigma$, write $\mathrm{im}(U^{\mathbb{C}})$ for the full subcategory of Σ consisting of the relational structures in the image of $U^{\mathbb{C}}$.

Theorem 2. *Assume that $\mathbb{C}$ and $\mathbb{C}^+$ are comonads on Σ and Σ^+, respectively, and also that*

1. *$\mathbb{C}^+$ restricts to finite σ^+-structures $\Sigma_f^+ \to \Sigma_f^+$, and*
2. *the embedding $J\colon \Sigma \to \Sigma^+$ and its left adjoint H restrict to $\Sigma_f \cap \mathrm{im}(U^{\mathbb{C}})$ and $\Sigma_f^+ \cap \mathrm{im}(U^{\mathbb{C}^+})$.*

Then, for any finite σ-structures A and B,

$$F^{\mathbb{C}^+}(J(A)) \cong F^{\mathbb{C}^+}(J(B)) \text{ if, and only if, } |\hom_{\Sigma_f}(C,A)| = |\hom_{\Sigma_f}(C,B)|$$

for every finite σ-structure C in $\mathrm{im}(U^{\mathbb{C}})$.

The proofs of Grohe's and Dvořák's theorems essentially reduce to showing that the assumptions of Theorem 2 are satisfied for the appropriate comonads. The "combinatorial core" of these results, requires a specific argument for each comonad and cannot be reduced to diagram chasing. In fact, verifying that the functor H restricts to $\Sigma_f^+ \cap \mathrm{im}(U^{\mathbb{C}^+}) \to \Sigma_f \cap \mathrm{im}(U^{\mathbb{C}})$, as required in 2, corresponds to checking that the operation $D \mapsto H(D)$ does not increase the tree-depth or tree-width.

We also apply the same machinery to another game comonad: the *graded modal comonad* $\mathbb{M}_k$ (cf. [3]). This gives a new Lovász-style result for pointed Kripke structures, relating homomorphism counts from synchronization trees of bounded height to the equivalence in graded modal logic.

In order to prove Theorem 2 we need the following corollary of Theorem 1, which is a direct consequence of the fact that the forgetful functor $\mathrm{EM}(\mathbb{C}) \to \mathscr{A}$, for a comonad $\mathbb{C}$ on $\mathscr{A}$, creates colimits and isomorphisms and also that any cocomplete category that is well-copowered admits a proper factorisation system.

Lemma 3. *Let $\mathbb{C}$ be any comonad on Σ. Then $\mathrm{EM}_f(\mathbb{C})$, the category of finite coalgebras for $\mathbb{C}$, is combinatorial.*

In fact, Lemma 3 holds in greater generality. The same proof would go through for any comonad $\mathbb{C}$ on a cocomplete and well-copowered category $\mathscr{A}$ with a locally finite full subcategory $\mathscr{A}_f$, assuming $\mathscr{A}_f$ is closed under finite colimits in $\mathscr{A}$ and $b \in \mathscr{A}_f$ whenever $a \to b$ is an epimorphism in $\mathscr{A}$ with $a \in \mathscr{A}_f$.

Proof sketch of Theorem 2. By Lemma 3, $F^{\mathbb{C}^+}(J(A)) \cong F^{\mathbb{C}^+}(J(B))$ if, and only if, $|\hom(\gamma, F^{\mathbb{C}^+}(J(A)))|$ is equal to $|\hom(\gamma, F^{\mathbb{C}^+}(J(B)))|$, for all finite coalgebras $\gamma\colon D \to \mathbb{C}^+(D)$. However, by $U^{\mathbb{C}^+} \dashv F^{\mathbb{C}^+}$, assumption 2 of the theorem and the fact that J is full and faithful, the right-hand-side of the last equivalence reduces to $|\hom_{\Sigma_f}(C,A)| = |\hom_{\Sigma_f}(C,B)|$, for every finite σ-structure C in $\mathrm{im}(U^{\mathbb{C}})$. □

3 Outlook

The power of our technique lies in the generality of our approach. Our method lays a pathway to discovering more Lovász-type theorems. In particular, any comonad on the category of σ-structures that satisfies the conditions of Theorem 2 will yield a Lovász-type theorem. The natural next step to test this theory is to apply our results to the game comonads introduced in [2] and [5].

References

[1] S. Abramsky, A. Dawar & P. Wang (2017): *The pebbling comonad in Finite Model Theory*. In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS*, pp. 1–12.

[2] S. Abramsky & D. Marsden (2020): *Comonadic semantics for guarded fragments*. Preprint available at arXiv:2008.11094.

[3] S. Abramsky & N. Shah (2018): *Relating Structure and Power: Comonadic Semantics for Computational Resources*. In: *27th EACSL Annual Conference on Computer Science Logic, CSL*, pp. 2:1–2:17.

[4] J.-Y. Cai, M. Fürer & N. Immerman (1992): *An Optimal Lower Bound on the Number of Variables for Graph Identification*. *Combinatorica* 12(4), pp. 389–410.

[5] A. Ó Conghaile & A. Dawar (2021): *Game Comonads & Generalised Quantifiers*. In: *29th EACSL Annual Conference on Computer Science Logic, CSL*, pp. 16:1–16:17.

[6] Z. Dvořák (2010): *On recognizing graphs by numbers of homomorphisms*. *Journal of Graph Theory* 64(4), pp. 330–342.

[7] M. Grohe (2020): *Counting Bounded Tree Depth Homomorphisms*. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, pp. 507–520.

[8] J. Isbell (1991): *Some inequalities in hom sets*. *Journal of Pure and Applied Algebra* 76(1), pp. 87–110.

[9] A. Joyal (1971): *Polyadic spaces and elementary theories*. *Notices of the AMS* 18(3), p. 563. Preliminary report.

[10] L. Lovász (1967): *Operations with structures*. *Acta Math. Acad. Sci. Hungar.* 18, pp. 321–328.

[11] L. Lovász (1972): *Direct product in locally finite categories*. *Acta Math. Acad. Sci. Hungar.* 23, pp. 319–322.

[12] A. Pultr (1973): *Isomorphism types of objects in categories determined by numbers of morphisms*. *Acta Scientiarum Mathematicarum* 35, pp. 155–160.